A.P. Calculus AB Test Five
Section Two
Free-Response
Calculators Allowed

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\, dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

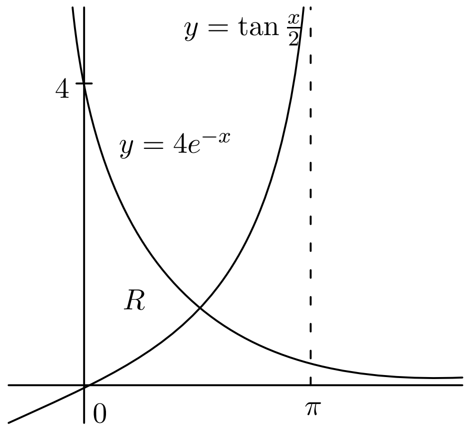
NAME:

Solution:

Multiple-Choice Answers

1	E
2	C
3	E
4	C
5	D
6	A
7	E
8	A
9	C
10	D
11	C
12	D
13	D
14	B
15	C

Free-response questions begin on the next page.



1. Let R be the region in the first quadrant enclosed by the graphs of $y = 4e^{-x}$, $y = \tan\left(\frac{x}{2}\right)$, and the y-axis, as shown in the figure above.

(a) Find the area of region R.

Solution: First, we find the x-coordinates of the intersection points of the two graphs. Set them equal and solve, using your calculator:

$$4e^{-x} = \tan\left(\frac{x}{2}\right)$$
$$x = 1.4786108$$

Let $a = 1.4786108$. Thus, the area A is

$$A = \int_0^a \left(4e^{-x} - \tan\left(\frac{x}{2}\right)\right)\, dx = 2.483$$

This part is worth 2 points:

1: integrand
1: answer

(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

Solution: The volume V is

$$V = \pi \int_0^a \left[\left(4e^{-x}\right)^2 - \left(\tan\left(\frac{x}{2}\right)\right)^2 \right] \, dx = 7.239\pi = 22.743$$

This part is worth 3 points:

2: integrand
  −1 for reversal, no constant, or other errors
1: answer

(c) The region R is the base of a solid, For this solid, each cross-section perpendicular to the x-axis is a semicircle. Find the volume of this solid.

Solution: Since the diameter is in R, the length of the radius is $\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]$. The area of a semicircle with radius r is $A = \frac{1}{2}\pi r^2$. Hence,

$$A = \frac{\pi}{2}\left(\frac{1}{2}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]\right)^2 = \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2.$$

Therefore, the volume V is

$$V = \int_0^a \frac{\pi}{8}\left[4e^{-x} - \tan\left(\frac{x}{2}\right)\right]^2 \, dx = 0.755\pi = 2.373$$

This part is worth 3 points:

2: integrand
  −1 incorrect, but has $4e^{-x} - \tan\left(\frac{x}{2}\right)$ as a factor
1: answer

There is also an additional overall point:

1: limits of integration in all parts

[This question is based on one from the 2003 AB Exam.]

 Continued on the next page.

2. Let $F(x) = \int_1^{2x} \sqrt{t^2 + t}\, dt$.

(a) Find $F'(x)$.

Solution: By the Fundamental Theorem of Calculus, we have

$$F'(x) = \sqrt{(2x)^2 + 2x} \cdot 2 = 2\sqrt{4x^2 + 2x}.$$

This part is worth 2 points:

1:	replaces t with $2x$
1:	finds derivative of $2x$
Note:	-1 each for antiderivative attempt, for writing $+C$, or for using constant value 1

(b) Find the domain of F.

Solution: We must have $t^2 + t \geq 0$; this happens when $t \leq -1$ or $t \geq 0$. Thus the area under the curve from 1 to $2x$ is undefined between -1 and 0. Therefore, the domain must be $x \geq 0$. This part is worth 2 points:

2: $x \geq 0$

 Continued on the next page.

(c) Find $\lim_{x \to 1/2} F(x)$.

Solution: Since $\frac{1}{2}$ is in the domain, we just plug in $\frac{1}{2}$ for x to get

$$\int_1^1 \sqrt{t^2+t}\,dt = 0$$

This part is worth 1 point:

1: answer

(d) Find the length of the curve $y = F(x)$ for $1 \le x \le 2$.

Solution: The length is given by

$$\begin{aligned} L &= \int_1^2 \sqrt{1+(F'(x))^2}\,dx \\ &= \int_1^2 \sqrt{1+(2\sqrt{4x^2+2x})^2}\,dx \\ &= \int_1^2 \sqrt{1+4(4x^2+2x)}\,dx \\ &= \int_1^2 \sqrt{16x^2+8x+1}\,dx \\ &= \int_1^2 \sqrt{(4x+1)^2}\,dx \\ &= \int_1^2 (4x+1)\,dx \\ &= 2x^2+x\Big|_1^2 \\ &= 7 \end{aligned}$$

This part is worth 4 points:

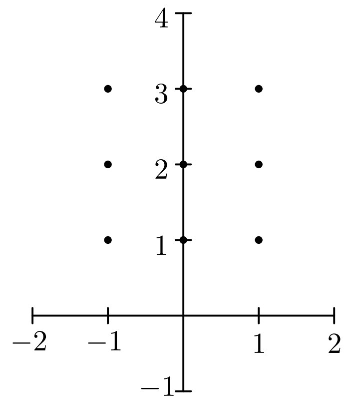
1: limits of integration
1: arc length form
1: uses F' from part (a)
1: answer

[This question is taken from the 1991 BC Exam.]

 Continued on the next page.

3. Consider the differential equation given by $\dfrac{dy}{dx} = \dfrac{xy}{2}$.

(a) On the graph below, sketch a slope field for the given differential equation at the nine points indicated.



Solution: You should have segments at nine points with negative - zero - positive slope left to right and increasing steepness bottom to top. This part is worth 1 point:

1: slope field

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

Solution: We use Euler's method twice:

$$f(0.1) \approx f(0) + f'(0)(0.1) = 3 + \frac{(0)(3)}{2}(0.1) = 3$$

$$f(0.2) \approx f(0.1) + f'(0.1)(0.1) = 3 + \frac{(0.1)(3)}{2}(0.1) = 3.015.$$

This part is worth 2 points:

1: Euler's method equations
1: answer; −1 if answer obtained without Euler's method

 Continued on the next page.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$. Use your solution to find $f(0.2)$.

Solution: To solve, we separate and integrate:

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy}{2} \\ \int \frac{dy}{y} &= \int \frac{x}{2}\,dx \\ \ln y &= \frac{1}{4}x^2 + C \\ y &= Ce^{x^2/4} \end{aligned}$$

With the initial condition, we find that $C = 3$, so the equation is $y = 3e^{x^2/4}$. Therefore, $f(0.2) = 3e^{0.4/4} = 3.030$. This part is worth 6 points:

1:	separation of variables
1:	antiderivative of dx term
1:	antiderivative of dy term
1:	solves for y
1:	value of C
1:	value of $f(0.2)$
Note:	maximum 4 of 6 points if no integration constant

[This question is taken from the 1998 BC Exam.]