A.P. Calculus BC Test One
Section Two
Free-Response
Calculators Allowed
Time—45 minutes
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y = x^2$ may not be written as `Y1=X^2`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

Solution:

Multiple-Choice Answers

1	D
2	E
3	D
4	B
5	C
6	B
7	B
8	C
9	D
10	B
11	B
12	C
13	E
14	D
15	A

Free-response questions begin on the next page.

1. Consider the function f defined below.

$$f(x) = \begin{cases} \dfrac{1}{x+2} & x < -2 \\ x^2 - 4 & x > -2. \end{cases}$$

(a) Find the zeros and any discontinuities of f, then draw the graph of f.

Solution: $\frac{1}{x+2}$ has no zeros for $x < -2$, but $x^2 - 4$ has the zero 2 for $x > -2$. There is an infinite discontinuity at $x = -2$, where $\frac{1}{x+2}$ has an asymptote. Your graph should reflect these facts, along with the fact that $\frac{1}{x+2}$ has a horizontal asymptote at $y = 0$ and that $x^2 - 4$ is a parabola whose vertex is at $(0, -4)$. This part is worth 4 points:

1: correct zero
1: correct type and location of discontinuity
2: correct graph; -1 for each missing item outlined above

 Continued on the next page.

(b) Determine $\lim_{x\to -2^-} f(x)$ and $\lim_{x\to -2^+} f(x)$. Does $\lim_{x\to -2} f(x)$ exist? Justify your answer.

Solution: From the left, $f(x) \to \infty$ as $x \to -2$ ("does not exist" is also acceptable); and from the right, $f(x) \to 0$ as $x \to -2$. The limit as $x \to -2$ does not exist since the left- and right-hand side limits are not equal. This part is worth 3 points:

1: correct answer as $x \to -2^-$
1: correct answer as $x \to -2^+$
1: correct asnwer as $x \to -2$ with justification

(c) Find the average rate of change of f from $x = -1$ to $x = 3$.

Solution: The average rate of change is

$$\frac{f(3) - f(-1)}{3 - (-1)} = \frac{5 - (-3)}{4} = \frac{8}{4} = 2.$$

This part is worth 2 points:

1: correct form
1: correct answer

 Continued on the next page.

2. Consider the function h defined below.

$$h(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ -1 & \text{if } x \text{ is not an integer} \end{cases}$$

(a) Sketch the graph of h.

Solution: You should have a series of points at $(a, 1)$ for every integer a, and a line at $y = -1$ with holes at integer x-values. This part is worth 2 points:

2: correct graph; -1 for each mistake

(b) What is $\lim_{x \to 3} h(x)$? What is $\lim_{x \to 3.5} h(x)$? For what numbers a does $\lim_{x \to a} h(x)$ exist?

Solution: $\lim_{x \to 3} h(x) = -1$; $\lim_{x \to 3.5} h(x) = -1$; the limit exists for all real numbers a. This part is worth 3 points:

1: correct limit as $x \to 3$
1: correct limit as $x \to 3.5$
1: correct limit as $x \to a$

 Continued on the next page.

(c) Now consider the function $k(x) = |x| - x$. Sketch the graph of this function.

Solution: This is basically a piecewise function defined as

$$k(x) = \begin{cases} 2x & x < 0 \\ 0 & x > 0 \end{cases}$$

This part is worth 1 point:

1: correct graph

(d) What is $\lim_{x \to 3} k(x)$? What is $\lim_{x \to -3} k(x)$? For what numbers a does $\lim_{x \to a} k(x)$ exist?

Solution: $\lim_{x \to 3} k(x) = 0$; $\lim_{x \to -3} k(x) = 6$; the limit exists for all real numbers a. This part is worth 3 points:

1: correct limit as $x \to 3$
1: correct limit as $x \to -3$
1: correct limit as $x \to a$

 Continued on the next page.

3. Consider the function $F(x) = a^{1/x}$ where a is a positive real number.

(a) What is the domain of F ? What are the zeros of F ?

Solution: The domain $D = (-\infty, 0) \cup (0, \infty)$ since zero makes the exponent undefined; and there are no zeros. This part is worth 2 points:

1: correct domain
1: correct zero

(b) Find $\lim_{x \to \infty} F(x)$ and $\lim_{x \to -\infty} F(x)$.

Solution: As $x \to \infty$, $F \to 1$. As $x \to -\infty$, $F \to 1$. This is true since $\frac{1}{x} \to 0$ as $x \to \pm\infty$, and so $a^{1/x} \to a^0 = 1$. This part is worth 2 points:

1: correct answer as $x \to \infty$
1: correct answer as $x \to -\infty$

 Continued on the next page.

(c) Does $\lim\limits_{x\to 0} F(x)$ exist? If so, find its value; if not, explain why not.
Does $\lim\limits_{x\to a} F(x)$ exist? If so, find its value; if not, explain why not.

> **Solution:** $\lim\limits_{x\to 0} F(x)$ does not exist since the left-hand side goes to zero and the right-hand side goes to ∞. The second limit is striaghtforward—$\lim\limits_{x\to a} F(x) = a^{1/a}$. This part is worth 2 points:
>
> 1: correct answer as $x \to 0$ with justification
> 1: correct answer as $x \to a$

(d) Find the value of a so that $F(3) = 8$. Then draw the graph of F using the value of a that you found.

> **Solution:** Solve $a^{1/3} = 8$ to get $a = 2$. Then graph $F(x) = 2^{1/x}$. This part is worth 3 points:
>
> 1: correct value of a
> 2: correct graph using your value of a