A.P. Calculus BC Test Two
Section Two
Free-Response
No Calculators

Time—40 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y = x^2$ may not be written as `Y1=X^2`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

Solution:

Multiple-Choice Answers

1	E
2	D
3	D
4	D
5	D
6	A
7	D
8	B
9	D
10	B
11	C
12	C
13	D
14	C
15	B

Free-response questions begin on the next page.

1. Let $f(x) = \sqrt{1 - \sin x}$.

(a) What is the domain of f ?

Solution: Since $-1 \leq \sin x \leq 1$, the domain is all real numbers. This part is worth 1 point:

1: domain

(b) Find $f'(x)$.

Solution:

$$f'(x) = \frac{1}{2}(1 - \sin x)^{-1/2}(-\cos x) = \frac{-\cos x}{2\sqrt{1 - \sin x}}.$$

This part is worth 2 points:

2: derivative

 Continued on the next page.

(c) What is the domain of f' ?

Solution: Here, $1 - \sin x \neq 0$, or $\sin x \neq 1$. This implies that x cannot be equal to $..., -7\pi/2, -3\pi/2, \pi/2, 5\pi/2, ...$, or, the domain is all real numbers except x such that $x = \frac{\pi}{2} + 2k\pi$ for integer k. This part is worth 3 points:

1: recognizing that $\sin x \neq 1$
2: domain

(d) Write an equation for the line tangent to the graph of f at $x = 0$.

Solution: The slope is $f'(0) = -\frac{1}{2}$ and at $x = 0$, the curve is at the point $(0, 1)$. Hence, the tangent is

$$-\frac{1}{2}(x - 0) = y - 1$$

or

$$y = -\frac{1}{2}x + 1.$$

This part is worth 3 points:

1: value of $f'(0)$
1: point of tangency
1: tangnet line equation

[This question is taken from the 1987 AB Exam.]

 Continued on the next page.

2. Consider the curve $y^3 + 7x^2 = x^3$.

(a) Write a general expression for the slope of the curve.

Solution: Implicit differentiation gives

$$\begin{aligned} 3y^2y' + 14x &= 3x^2 \\ 3y^2y' &= 3x^2 - 14x \\ y' &= \frac{3x^2 - 14x}{3y^2} \end{aligned}$$

This part is worth 2 points:

1: differentiation
1: solves for y'

(b) Find the equation of the tangent line to the curve at $x = 8$.

Solution: At $x = 8$, $y = 4$. Thus the slope is

$$y' = \frac{3(8^2) - 14(8)}{3(4^2)} = \frac{192 - 112}{48} = \frac{80}{64} = \frac{5}{4}.$$

Therefore, the tangent line is

$$\frac{5}{4}(x - 8) = y - 4$$

or

$$y = \frac{5}{4}x - 6.$$

This part is worth 3 points:

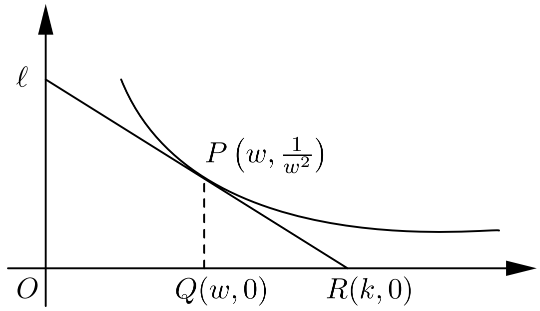
1: value of y
1: value of y'
1: tangent line equation

 Continued on the next page.

(c) Find all points on the curve where tangent lines are horizontal or vertical.

Solution: Horizontal tangents occur when $y' = 0$. Set the numerator of the derivative equal to zero to get $x(3x-14) = 0$, or $x = 0$ and $x = \frac{14}{3}$. Vertical tangents occur when the derivative is undefined. Set the denominator of the derivative equal to zero to get $y = 0$. We are asked for points, so when $x = 0$, $y = 0$; when $x = \frac{14}{3}$, $y = -\frac{7}{3}\sqrt[3]{4}$; and when $y = 0$, $x = 0$ or $x = 7$. But this implies that there is a horizontal tangent and a vertical tangent at $(0, 0)$! This cannot happen—the only conclusion is that there is a cusp at $(0, 0)$, so that the only vertical tangent is at $(7, 0)$ and the only horizontal tangent is at $\left(\frac{14}{3}, -\frac{7}{3}\sqrt[3]{4}\right)$. This part is worth 4 points:

1: sets $dy/dx = 0$
1: determines when dy/dx is undefined
1: rules out $(0, 0)$ as a point of tangency
1: both points



3. In the figure above, line ℓ is tangent to the graph of $y = \dfrac{1}{x^2}$ at point $P\left(w, \frac{1}{w^2}\right)$, where $w > 0$. Point Q has coordinates $(w, 0)$. Line ℓ crosses the x-axis at the point $R(k, 0)$.

(a) Find the value of k when $w = 3$.

Solution: Since $y' = -\frac{2}{x^3}$, we have $y'(3) = -\frac{2}{27}$. Thus, line ℓ goes through $(3, \frac{1}{9})$ and $(k, 0)$ with slope $-\frac{2}{27}$:

$$\frac{0 - \frac{1}{9}}{k - 3} = -\frac{2}{27}$$
$$0 - \frac{1}{9} = -\frac{2}{27}(k - 3)$$
$$k = \frac{9}{2}$$

This part is worth 2 points:

1: value of $y'(3)$
1: value of k

(b) For all $w > 0$, find k in terms of w.

Solution: Line ℓ goes through $(w, \frac{1}{w^2})$ and $(k, 0)$ with slope $-\frac{2}{w^3}$:

$$\frac{0 - \frac{1}{w^2}}{k - w} = -\frac{2}{w^3}$$
$$0 - \frac{1}{w^2} = -\frac{2}{w^3}(k - w)$$
$$k = \frac{3}{2}w$$

This part is worth 2 points:

1: equation relating w and k using slopes
1: answer

 Continued on the next page.

(c) Suppose w is increasing at a constant rate of 7 units per second. When $w = 5$, what is the rate of change of k with respect to time?

> **Solution:** Implicit differentiation gives
>
> $$\frac{dk}{dt} = \frac{3}{2}\frac{dw}{dt} = \frac{3}{2} \cdot 7 = \frac{21}{2}.$$
>
> This part is worth 1 point:
>
> 1: answer

(d) Suppose w is increasing at a constant rate of 7 units per second. What is the rate of change in the area of triangle PQR with respect to time?

> **Solution:** The base of the triangle is $k - w$ and the height is $\frac{1}{w^2}$; recalling from part (b) that $k = \frac{3}{2}w$, we have the area as
>
> $$A = \frac{1}{2}(k - w)\frac{1}{w^2} = \frac{1}{2}\left(\frac{3}{2}w - w\right)\frac{1}{w^2} = \frac{1}{4w}.$$
>
> Differentiating implicitly gives
>
> $$\frac{dA}{dt} = -\frac{1}{4w^2}\frac{dw}{dt}$$
> $$\frac{dA}{dt} = -\frac{1}{4(5^2)} \cdot 7$$
> $$\frac{dA}{dt} = -\frac{1}{100} \cdot 7 = -0.07$$
>
> This part is worth 4 points:
>
> 1: area formula in terms of w and/or k
> 1: differentiation
> 1: uses values of w and dw/dt
> 1: answer
>
> *[This question is taken from the 1999 AB Exam.]*