A.P. Calculus BC Test Three
Section Two
Free-Response
No Calculators

Time—45 minutes

Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $y'(2) = 3$ may not be written as `nDeriv(Y1,X,2)=3`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

Solution:

Multiple-Choice Answers

1	B
2	C
3	C
4	D
5	E
6	D
7	C
8	D
9	A
10	A
11	A
12	E
13	B
14	D
15	C

Free-response questions begin on the next page.

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1. A person has 340 yards of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide, and the other a square. The field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.

(a) If x is the width of the rectangular field, what are the maximum and minimum possible values of x ?

> **Solution:** Let y be the width of the square field. Then the rectangle has perimeter $x + x + 2x + 2x = 6x$ and the square has perimeter $4y$; hence, $6x + 4y = 340$. Since $y^2 \geq 100$ and $x^2 \geq 800$, we have $y \geq 10$ and $x \geq 20$. Therefore,
>
> $$\begin{aligned} 4y &= 340 - 6x \\ y &= 85 - \frac{3}{2}x \end{aligned}$$
>
> and so
>
> $$\begin{aligned} y &\geq 10 \\ 85 - \frac{3}{2}x &\geq 10 \\ -\frac{3}{2}x &\geq -75 \\ x &\leq 50 \end{aligned}$$
>
> and we have that the maximum of x is 50 and the minimum of x is 20. This part is worth 4 points:
>
> 1: equation relating x and y
> 1: inequalities in x and y
> 1: maximum x
> 1: minimum x

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(b) What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.

Solution: Let A be the total area. Then

$$A = 2x^2 + y^2 = 2x^2 + \left(85 - \frac{3}{2}x\right)^2 = \frac{17}{4}x^2 - 255x + 7225.$$

We seek the maximum, so we find critical points of the derivative:

$$A' = \frac{17}{2}x - 255$$

Setting this equal to zero and solving gives $x = 30$; however, the second derivative is positive, indicating that $x = 30$ is a minimum. Thus, the maxima occur at the endpoints—either $x = 20$ or $x = 50$. (Alternatively, one could say that $A(x)$ is a positive parabola which only has a minimum—again indicating the maximum occurs at the endpoints.)

At $x = 20$, $y = 55$, so that $A = 3825$; at $x = 50$, $y = 10$, so that $A = 5100$. Thus, 5100 is the greatest number of square yards that can be enclosed. This part is worth 5 points:

1: area equation
2: derivative, or uses parabola argument
1: rules out $x = 30$ as a maximum
1: finds maximum value

[This question is taken from the 1972 AB Exam.]

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2. A function f is continuous on the closed interval $[-3, 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The function f' and f'' have the properties given in the table below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	positive	fails to exist	negative	0	negative
$f''(x)$	positive	fails to exist	positive	0	negative

(a) What are the x-coordinates of all absolute maximum and absolute minimum points of f on the interval $[-3, 3]$? Justify your answer.

Solution: The absolute maximum occurs at $x = -1$ because f is increasing on the interval $[-3, -1]$ and decreasing on the interval $[-1, 3]$. The absolute minimum must occur at $x = 1$ or at an endpoint. However, f is decreasing on the interval $[-1, 3]$; therefore, the absolute minimum is at an endpoint. Since $f(-3) = 4 > 1 = f(3)$, the absolute minimum is at $x = 3$. This part is worth 4 points:

1: absolute maximum
1: absolute minimum
2: justification

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(b) What are the x-coordinates of all points of inflection of f on the interval $[-3, 3]$? Justify your answer.

Solution: There is an inflection point at $x = 1$ because the graph changes from concave up to concave down (or f'' changes from positive to negative) there. This part is worth 2 points:

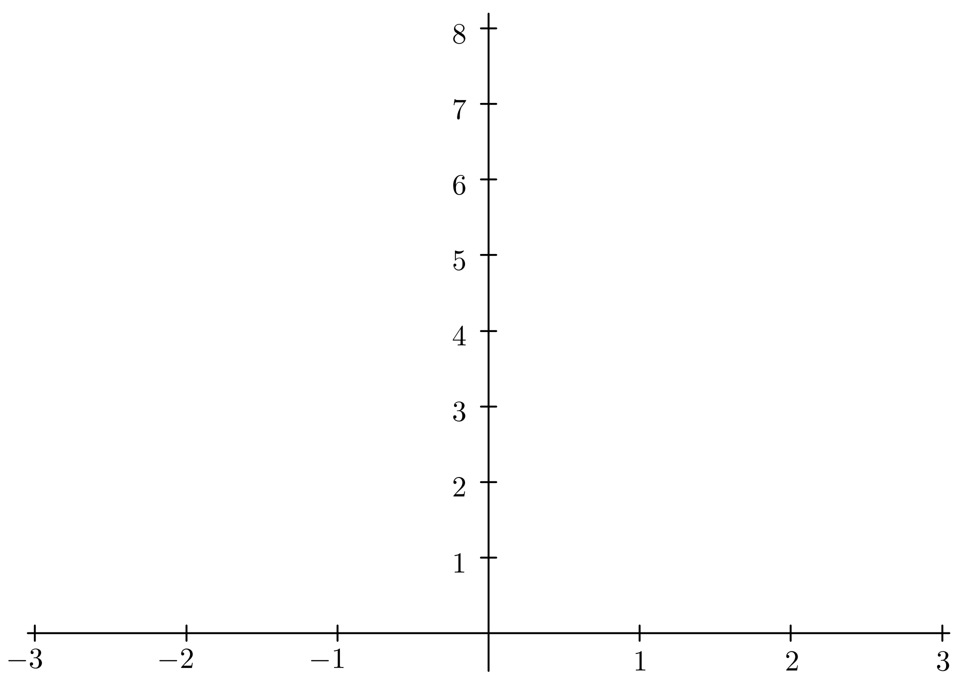
1: inflection point
1: justification

(c) On the axes provided, sketch a graph that satisfies the given properties of f.

Solution: This part is worth 3 points:

1: endpoints at $(-3, 4)$ and $(3, 1)$
1: increasing/decreasing, and max at $x = -1$
1: concavity, and inflection at $x = 1$

[This question is taken from the 1984 AB Exam.]



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3. Let $F(x) = \dfrac{x}{1+x^2}$.

(a) Determine if F is an odd or even function, then find all asymptotes of F.

Solution: Since

$$F(-x) = \frac{-x}{1+(-x)^2} = -\frac{x}{1+x^2} = -F(x)$$

we have that F is an odd function. Since the denominator of F is never zero, there are no vertical asymptotes. The horizontal asymptote is the line $y = 0$. This part is worth 2 points:

1: determines F is odd
1: asymptote at $y = 0$

(b) Give the x-coordinates of all local maximum and minimum points of F.

Solution: The derivative is

$$F'(x) = \frac{1+x^2 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2};$$

setting $F'(x) = 0$ and solving gives $x = \pm 1$. Since $F' < 0$ for $x > 1$, and since F is odd, we must have a maximum at $x = 1$ and a minimum at $x = -1$. This part is worth 3 points:

1: derivative
1: $x = 1$ is a maximum with justification
1: $x = -1$ is a minimum with justification

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(c) Find the x-coordinates of all inflection points of f.

Solution: We need the second derivative:

$$F''(x) = \frac{(-2x)(1+x^2)^2 - (1-x^2)(4x)(1+x^2)}{(1+x^2)^4}$$
$$= \frac{(1+x^2)(-2x-2x^3-4x+4x^3)}{(1+x^2)^4} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

Setting $F''(x) = 0$ and solving gives $x = -\sqrt{3}, 0, \sqrt{3}$ This part is worth 2 points:

1: second derivative
1: all three inflection points

(d) Find the equation of the line tangent to the graph of F at $x = 0$ and use it to approximate $F(0.1)$.

Solution: The slope of the tangent is $F'(0) = 1$ and the tangent passes through the origin; therefore, the tangent line is $y = x$. Hence, $F(0.1) \approx 0.1$. This part is worth 2 points:

1: tangent line
1: approximation of $F(0.1)$.