A.P. Calculus BC Test Four
Section Two
Free-Response
Calculators Allowed
Time—45 minutes
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

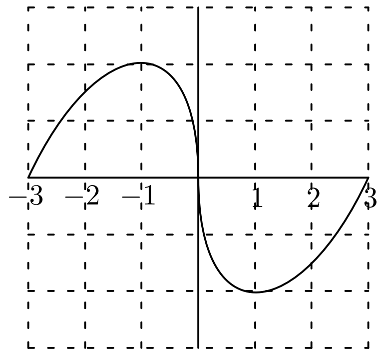
NAME:

Solution:

Multiple-Choice Answers

1	E
2	C
3	C
4	E
5	B
6	E
7	D
8	D
9	C
10	C
11	C
12	D
13	D
14	A
15	C

Free-response questions begin on the next page.



Graph of f

1. Let $G(x) = \displaystyle\int_{-3}^{x} f(t)\,dt$, where the graph of the function f is given above for $-3 \le t \le 3$.

(a) Evaluate $G(3)$ and $G(-3)$.

Solution: Since f is odd, $\displaystyle\int_{-3}^{3} f\ dt = G(3) = 0$. Obviously, $\displaystyle\int_{-3}^{-3} f\ dt = G(-3) = 0$. This part is worth 2 points:

1: value of $G(3)$
1: value of $G(-3)$

(b) On what interval is G increasing? Where does G have a maximum value?

Solution: Since $G'(x) = f(x)$ by the Fundamental Theorem, setting $G'(x) = f(x) = 0$ and solving gives critical points of $x = -3, 0, 3$. By the graph, $G'(x) > 0$ for $-3 < x < 0$ and $G'(x) < 0$ for $0 < x < 3$; hence, G is increasing for $-3 < x < 0$. The maximum value occurs at $x = 0$ since there is a sign change there. This part is worth 3 points:

1: uses Fundamental Theorem
1: increasing interval
1: maximum

(c) On what interval is G concave down? Justify your answer.

Solution: For concave down, we want $G''(x) < 0$, which implies $f'(x) < 0$, which implies f is decreasing. Since f is decreasing for $-1 < x < 1$, we have G is concave down for $-1 < x < 1$. This part is worth 2 points:

1: answer
1: justification

(d) Sketch a graph of G.

Solution: Your graph should begin at $(-3, 0)$, end at $(3, 0)$, and have a maximum at $(0, 4)$; be increasing for $-3 < x < 0$ and decreasing for $0 < x < 3$; be concave down for $-1 < x < 1$ and concave up for $-3 < x < -1$ and $1 < x < 3$. This part is worth 2 points:

1: beginning, ending, max points
1: increasing, decreasing, concavity

2. The region R_1 is bounded above by $g(x) = 2^x$ and below by $f(x) = x^2$, while the region R_2 is bounded above by $f(x) = x^2$ and bounded below by $g(x) = 2^x$.

(a) Find the x- and y-coordinates of the three points of intersection of the graphs of f and g.

> **Solution:** Simply use the calculator to find $(2, 4)$, $(4, 16)$, and $(-0.767, 0.588)$. This part is worth 1 point:
>
> 1: all three points

(b) Without using absolute value, set up an expression involving one or more integrals that gives the total area enclosed by the graphs of f and g. Do not evaluate.

> **Solution:**
> $$A = \int_{-0.767}^{2} (2^x - x^2)\ dx + \int_{2}^{4} (x^2 - 2^x)\ dx.$$
> This part is worth 4 points:
>
> 2: limits of integration
> 2: integrands

 Continued on the next page.

(c) Without using absolute value, set up an expression involving one or more integrals that gives the volume of the solid generated by revolving the region R_1 about the line $y = 5$. Do not evaluate.

Solution:

$$V = \pi \int_{-0.767}^{2} \left((x^2 - 5)^2 - (2^x - 5)^2 \right) \, dx.$$

This part is worth 4 points:

1: constant
1: limits of integration
2: integrand

[This question is taken from the 1995 BC Exam.]

 Continued on the next page.

3. Let f be the function given by $f(x) = \sqrt{x-3}$.

(a) Find the area of the region R enclosed by the graph of f, the x-axis, and the vertical line $x = 6$.

Solution:

$$R = \int_3^6 \sqrt{x-3}\, dx = 3.464$$

This part is worth 2 points:

1: limits of integration
1: answer

(b) Rather than using the line $x = 6$ as in part (a), consider the line $x = w$, where $w > 3$. Let $A(w)$ be the area of the region enclosed by the graph of f, the x-axis, and the vertical line $x = w$. Write an integral expression for $A(w)$.

Solution:

$$A(w) = \int_3^w \sqrt{x-3}\, dx.$$

This part is worth 2 points:

1: limits of integration in terms of w
1: integrand

Continued on the next page.

(c) Let $A(w)$ be as described in part (b). Find the rate of change of A with respect to w when $w = 6$. Justify your answer.

Solution: We find the derivative of $A(w)$ by the Fundamental Theorem:

$$A'(w) = \frac{d}{dw} \int_3^w \sqrt{x-3}\, dx = \sqrt{w-3}.$$

At $w = 6$, $A'(6) = \sqrt{6-3} = \sqrt{3}$. This part is worth 2 points:

1: non-integral expression for A'
1: value of $A'(6)$

(d) Let $V(w)$ be the volume of the solid that results from revolving the region described in part (b) about the x-axis. Find an integral expression for $V(w)$ then use it to evaluate $V(6)$.

Solution: The volume is

$$V(w) = \pi \int_3^w \sqrt{x-3}^2\, dx = \pi \int_3^w (x-3)\, dx,$$

and at $w = 6$,

$$V(6) = \pi \int_3^6 (x-3)\, dx = \frac{9}{2}\pi = 14.137.$$

This part is worth 3 points:

1: limits of integration in terms of w
1: constant and integrand
1: value of $V(6)$

[This question is taken from the 1997 AB Exam.]