A.P. Calculus BC Test Six
Section Two
Free-Response
Calculators Allowed
Time—45 minutes
Number of Questions—3

Each of the three questions is worth 9 points. The maximum possible points earned on this section is 27, which represents 50% of the total test score. There is no penalty for guessing.

- SHOW ALL YOUR WORK. You will be graded on the methods you use as well as the accuracy of your answers. Correct answers without supporting work may not receive full credit.
- Write all work for each problem in the space provided. Be sure to write clearly and legibly. Erased or crossed out work will not be graded.
- Justifications require that you give mathematical (non-calculator) reasons and that you clearly identify functions, graphs, tables, or other objects that you use.
- You are permitted to use your calculator to solve an equation or graph a function without showing work. However, you must clearly indicate the setup of your problem.
- Your work must be expressed in mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\, dx$ may not be written as `fnInt(X^2,X,1,5)`.
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

Good Luck!

NAME:

Solution:

Multiple-Choice Answers

1	C
2	E
3	D
4	A
5	D
6	A
7	D
8	C
9	A
10	D
11	B
12	A
13	D
14	A
15	D

Free-response questions begin on the next page.

1. Let

$$P_4(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$$

be a Taylor polynomial of order four for the function f at $x = 4$. Assume f has derivatives of all orders for all real numbers.

(a) Find $f(4)$ and $f''(4)$.

Solution: Clearly, $f(4) = 7$ and $f''(4) = 10$. This part is worth 2 points:

1: value of $f(4)$
1: value of $f''(4)$

(b) Write the second-degree Taylor polynomial for f' about 4 and use it to approximate $f'(4.3)$.

Solution:

$$f'(x) \approx -3 + 10(x-4) - 6(x-4)^2$$

which gives $f(4.3) \approx -0.54$. This part is worth 3 points:

2: Taylor polynomial for $f'(x)$
-1 for each incorrect or extra term, or $+\cdots$
1: value of $f'(4.3)$

 Continued on the next page.

(c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)\, dt$ at $x = 4$.

Solution:

$$g(x) \approx 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$$

This part is worth 3 points:

3: Taylor polynomial for $g(x)$
-1 for each incorrect or extra term or $+\cdots$

(d) Can the exact value of $f(3)$ be determined from the information given? Justify your answer.

Solution: The exact value cannot be determined because we are only given values of f at $x = 4$. This part is worth 1 point:

1: answer with justification

[This question is taken from the 1997 BC Exam.]

2. Two particles move in the xy-plane. For time $t \geq 0$, the position of particle A is given by $x = t - 2$ and $y = (t-2)^3$, and the position of particle B is given by $x = \frac{3t}{2} - 4$ and $y = \frac{3t}{2} + 2$.

(a) Find the velocity vector for each particle at time $t = 3$.

Solution: Particle A:

$$\mathbf{v}(t) = \left\langle 1,\ 3(t-2)^2 \right\rangle$$
$$\mathbf{v}(3) = \langle 1,\ 3 \rangle$$

Particle B:

$$\mathbf{v}(t) = \left\langle \frac{3}{2},\ \frac{3}{2} \right\rangle = \mathbf{v}(3)$$

This part is worth 3 points:

1: expression for velocity of A
1: value of velocity of A at $t = 3$
1: velocity of B

(b) Set up an integral expression that gives the distance traveled by particle A from $t = 0$ to $t = 3$. Do not evaluate.

Solution: This is just arc length.

$$\int_0^3 \sqrt{1 + [3(t-2)^2]^2}\ dt$$

This part is worth 2 points:

2: integral

 Continued on the next page.

(c) Determine the exact time at which the particles collide; that is, when the particles are at the same point at the same time. Justify your answer.

Solution: The x-coordinates of A and B must be the same for collision; hence we solve:

$$\begin{aligned} t - 2 &= \frac{3t}{2} - 4 \\ 2t - 4 &= 3t - 8 \\ t &= 4 \end{aligned}$$

Now we check the y-coordinates at $t = 4$: the y value of A is $(4-2)^3 = 8$ and the y value of B is $\frac{3(4)}{2} + 2 = 8$. Therefore, the particles are at the same point $(2, 8)$ when $t = 4$. This part is worth 3 points:

1: sets either x or y coordinates equal
1: value $t = 4$
1: justification

(d) Sketch the paths of the particles A and B from $t = 0$ until they collide. Indicate the direction of each particle along its path.

Solution: The path of particle A is the graph of $y = x^3$ from $(-2, -8)$ to $(2, 8)$; that of particle B is the graph of $y = x + 4$ from $(-4, 2)$ to $(2, 8)$. This part is worth 1 point:

1: graphs with direction

[This question is taken from the 1995 BC Exam.]

3. Consider the family of polar curves defined by $r = 2 + \cos k\theta$, where k is a positive integer.

(a) Show that the area of the region enclosed by the curve does not depend on the value of k. What is the area?

Solution:

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi} (2+\cos k\theta)^2\ d\theta &= \frac{1}{2}\int_0^{2\pi} (4 + 4\cos k\theta + \cos^2 k\theta)\ d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \left(\frac{9}{2} + 4\cos k\theta + \frac{1}{2}\cos 2k\theta\right)\ d\theta \\ &= \frac{1}{2}\left(\frac{9}{2}\theta + \frac{4}{k}\sin k\theta + \frac{1}{4k}\sin 2k\theta\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{2}(9\pi) = \frac{9\pi}{2} \end{aligned}$$

Thus, the area is independent of the value of k. This part is worth 5 points:

1:	in the form $\frac{1}{2}\int r^2\ d\theta$
1:	limits of integration
2:	antiderivative
1:	answer
Note:	zero points if plugs in specific value of k

(b) Write an expression in terms of k and θ for the slope dy/dx of the curve.

Solution:

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-k\sin k\theta\sin\theta + (2+\cos k\theta)\cos\theta}{-k\sin k\theta\cos\theta - (2+\cos k\theta)\sin\theta}$$

This part is worth 2 points:

2: derivative in terms of k and θ

(c) If k is a multiple of 4, then find the value of dy/dx at $\theta = \pi/4$.

Solution: If k is a multiple of 4, then $k\theta$ is a multiple of π; this implies that $\sin k\theta = 0$ and $\cos k\theta = \pm 1$. Hence,

$$\frac{-k\sin k\theta\sin\theta + (2+\cos k\theta)\cos\theta}{-k\sin k\theta\cos\theta - (2+\cos k\theta)\sin\theta} = \frac{(2\pm 1)(\frac{\sqrt{2}}{2})}{-(2\pm 1)(\frac{\sqrt{2}}{2})} = -1$$

This part is worth 2 points:

2: value of dy/dx