The First Great Theorem

The History of Mathematics, Part 4

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Outline

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Babylonian

Clay tablet (*Plimpton 322*) from Babylon, c.2000 BC

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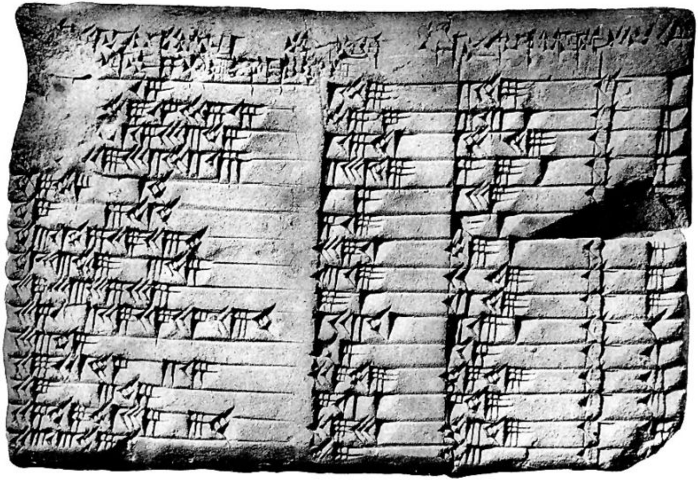
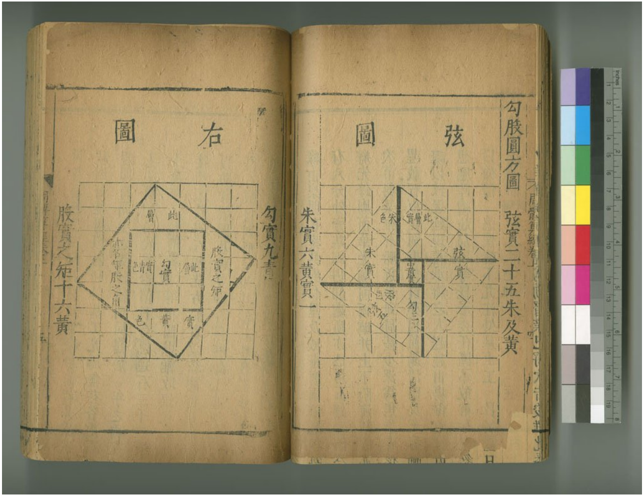


Second column: shorter leg
Third column: hypotenuse

Chinese

Called the “gougu” rule



1600 AD copy of a book written in 1500 BC

Egyptian

- Knew special right triangles
- Used ropes with knots equally spaced apart
- No formal rules for finding Pythagorean triples

Outline

Pythagoras

Pythagoras of Samos

570 BC-475 BC

"Do not say a little in many words but a great deal in a few."

Pythagoras

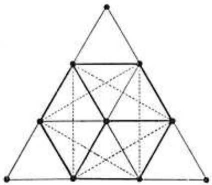
- Possibly studied under Thales; traveled to Egypt and possibly India
- Founded a school in Crotona with elaborate rites and beliefs; all discoveries made by students were credited to Pythagoras
- Possibly was the first to prove the Pythagorean Theorem
- Believed every soul is immortal and, upon death, enters into a new body
- Planets move according to mathematical equation and thus resonate to produce a symphony of music
- Conflict with supporters of democracy forced Pythagoras and his followers to flee Crotona

Pythagorean School

- Whole numbers rule the universe in all ways
- “Divine Number” bestowed blessings
- School emblem was a 5-pointed star

Pythagorean School

- Whole numbers rule the universe in all ways
- “Divine Number” bestowed blessings
- School emblem was a 5-pointed star
- Sacred symbol was a tetractys



Pythagorean School

- Women treated as equals
- "All things in common among friends"
- Two groups: *mathematikoi* (learners – rational, scientific) and *askousmatikoi* (listeners – mystic, religious).

Pythagorean School

Number beliefs: odds are masculine, evens feminine

1 The source of all things; the monad

2 The dyad; all matter

3 ideal; it has beginning, middle, end

4 seasons and elements

5 Marriage

6 Luck

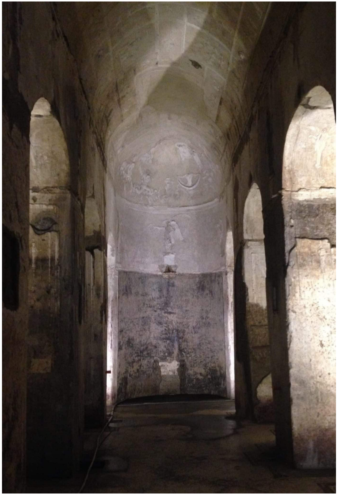
7 planets; strings on a lyre

10 the perfect number

Pythagorean Influence

- Was the first to call himself "philosopher"
- Ideas on mathematical perfection influenced Greek art
- Teachings were revived in first century BC (so-called "neopythagoreanism") and again in Middle Ages, influencing mathematicians in 1600s
- Transmigration of souls in Christianity
- Fictional portrayal in Ovid's *Metamorphoses* in which Pythagoras urged a strict vegetarian diet inspired the modern vegetarian movement – before "vegetarian" was used in 1840s, vegetarians were called "pythagoreans"

Neopythagorean Temple



Oldest known Pythagorean temple: first century BC, 40 feet below street level in outer Rome

Pythagoras

Statue on Isle of Samos, erected in 1988

Pythagoras

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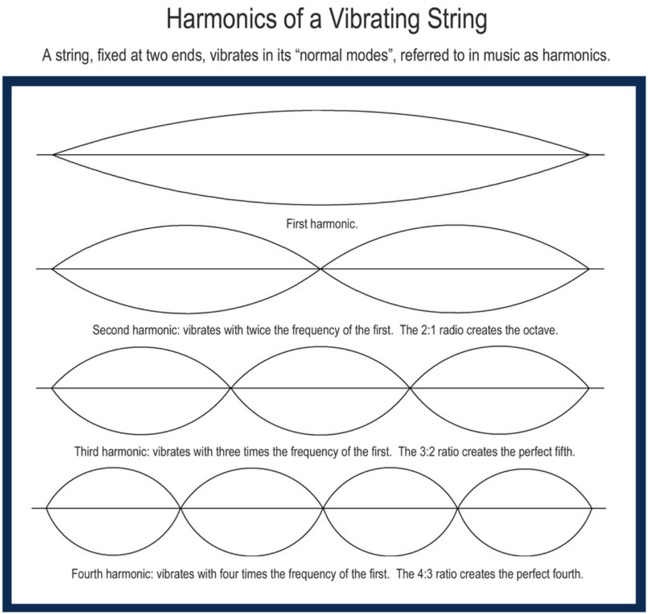
Harmonics

- Harmonies of notes between plucked strings occur when the lengths of strings are in whole number ratios
- The most harmonius interval between notes is the *octave*

Harmonics


Harmonics of a Vibrating String
A string, fixed at two ends, vibrates in its "normal modes", referred to in music as harmonics.
First harmonic.
Second harmonic: vibrates with twice the frequency of the first. The 2:1 radio creates the octave.
Third harmonic: vibrates with three times the frequency of the first. The 3:2 ratio creates the perfect fifth.
Fourth harmonic: vibrates with four times the frequency of the first. The 4:3 ratio creates the perfect fourth.

Harmonics

- When the fifth ($3 : 2$ ratio) and the fourth ($4 : 3$ ratio) are combined (added), they create the octave, since
$$\frac{3}{2} \times \frac{4}{3} = 2.$$
- These are natural steps smaller than the octave. What about the other steps of the 8-note scale?
- Add fifths! (As the Pythagoreans thought.)

Harmonics

- Two fifths: $\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}$. More than 2 octaves, so the pitch is raised by more than an octave
- Size of step over 1 octave is $\frac{9}{4} \div 2 = \frac{9}{8}$. Called a *second*.
- "Add" fifths by multiplying $3/2$ to itself, "subtract" octaves by dividing by 2, until the "difference" is less than an octave.
- 12 fifths is very close to 7 octaves:

$$\left(\frac{3}{2}\right)^{12} \div 2^7 = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1.0136$$

- This interval (1.0136) is called the *Pythagorean comma*, about $1/4$ of the smallest step in the scale.

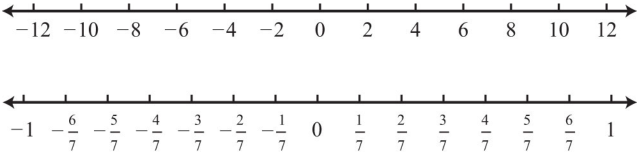
Harmonics

- Pythagoreans never found what number of factors of $3/2$ (fifths) exactly equals a number of factors of 2 (octaves).
- Modern adjustment: Build the octave from 12 equal parts (called a *semitone*).
- So 12 multiplcations are needed to make an octave: each semitone is $2^{1/12}$. Then $(2^{1/12})^7 \approx 1.49831$ is almost a perfect fifth
- One fret to the next on a guitar changes the length of the vibrating string by $2^{1/12}$

Outline

Commensurable Numbers

Any measure can be expressed as the ratio of two numbers; or, For any two measures, they may be expressed as integer multiples of a third segment.



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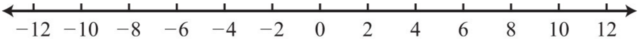
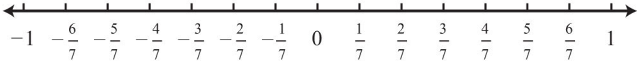
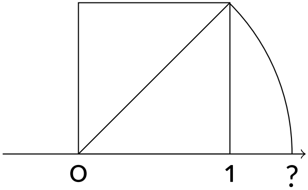
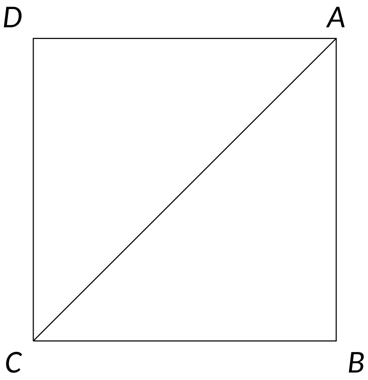






A Pythagorean Problem

Determine the ratio of the side of a square to its diagonal.



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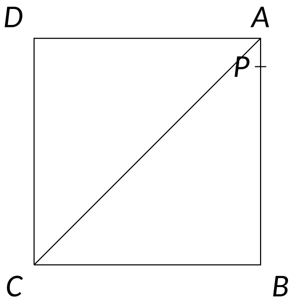
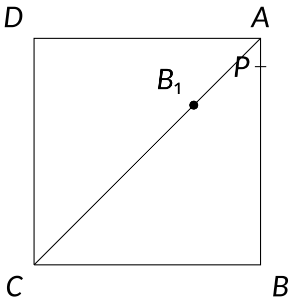


Suppose there is a segment *AP* that measures the side and the diagonal.

A Pythagorean Problem



Suppose there is a segment AP that measures the side and the diagonal.
On AC, mark off $CB_1 = AB$.

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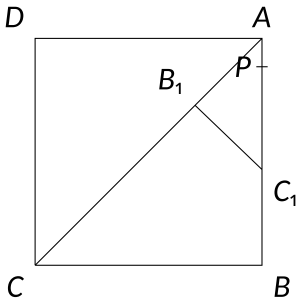

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Draw B_1C_1 perpendicular to CA.

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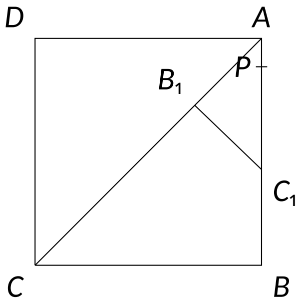

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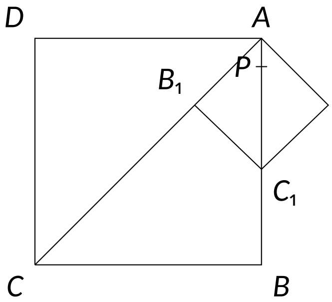

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On *AC*, mark off $CB_1 = AB$.
Draw B_1C_1 perpendicular to *CA*.
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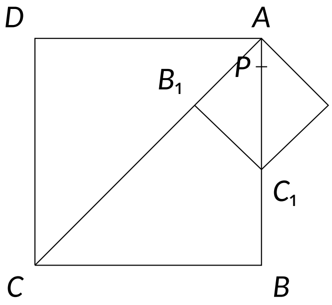
Hence $AC_1 = AB - AB_1$, and AC_1 and AB_1 are commensurable with *AP*.

A Pythagorean Problem



CA.
Then $C_1B = C_1B_1 = AB_1$.
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But AC_1 and AB_1 are a diagonal and side less than half the original.

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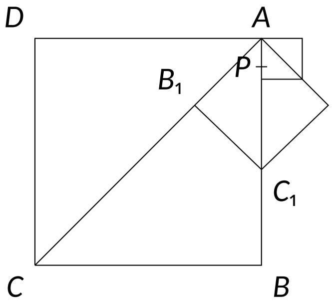
A Pythagorean Problem



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On *AC*, mark off $CB_1 = AB$.
Draw B_1C_1 perpendicular to *CA*.
Then $C_1B = C_1B_1 = AB_1$.
Hence $AC_1 = AB - AB_1$, and AC_1 and AB_1 are commensurable with *AP*.
But AC_1 and AB_1 are a diagonal and side less than half the original.
Now repeat the process.
Eventually we get a side and diagonal commensurable with *AP* but that is less than *AP*.

An Incommensurable

Result: A side and diagonal of a square are incommensurable.

Modern Result: $\sqrt{2}$ is irrational.

An Incommensurable

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Modern Result: $\sqrt{2}$ is irrational.

So that's why they couldn't fit an integer number of fifths in an integer number of octaves: $2^{1/12}$ is irrational!

Outline

Homework

- Last-Minute Problems, #2 – due February 8
- More on the Pythagorean Theorem; *Math Through the Ages*, Sketch 12

Next: Inconvenient Incorrigible Incomparable Incommensurables