Those Inconvenient Incorrigible Incomparable Incommensurables

The History of Mathematics, Part 5

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February 3, 2021

Outline

More and More Incommensurables

Three Ancient Problems

How Incommensurables Undermined Mathematics

How the Situation Was Resolved

Homework

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Three Ancient Problems

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How the Situation Was Resolved

Homework

Evidence of Irrationals before Greeks

Babylonian tablet c.1800 BC

Evidence of Irrationals before Greeks

Babylonian number across the middle: 1 ; 24 , 51 , 10

Evidence of Irrationals before Greeks

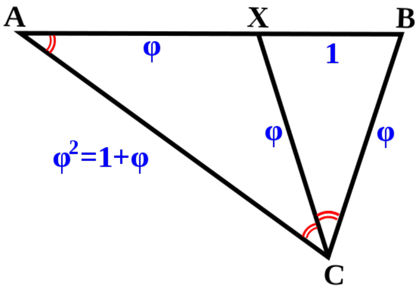
Babylonian number across the middle: 1 ; 24 , 51 , 10

$1\,;\,24\,,\,51\,,\,10 = 1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3} \approx 1.414212963$;
calculator gives $\sqrt{2} \approx 1.414213562$

Open the Floodgates

With the Pythagorean discovery that $\sqrt{2}$ is incommensurable, more incommensurable numbers were found…

The 36-36-72 Triangle



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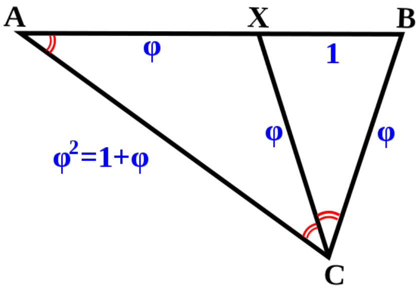

$\triangle ABC$ is similar to $\triangle CXB$. Thus,

$$\frac{XB}{AX} = \frac{AX}{AB} \quad \text{or} \quad \frac{1}{\phi} = \frac{\phi}{\phi + 1} \quad \text{or} \quad \phi + 1 = \phi^2.$$

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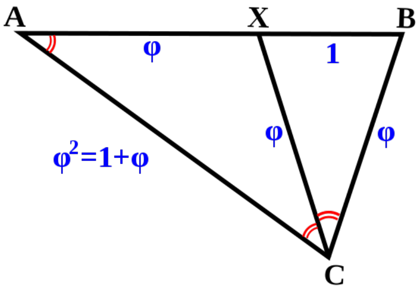

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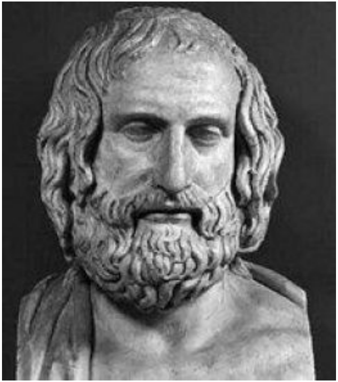
Solving for ϕ, we get $\phi = \dfrac{1 + \sqrt{5}}{2}$.

Outline

Squaring the Circle

- Find a square whose area is the same as a circle
- Impossible since π is incommensurable!

Squaring the Circle



Anaxagoras

500 BC-427 BC

"Men would live exceedingly quiet if these two words, mine and thine, were taken away."

Squaring the Circle

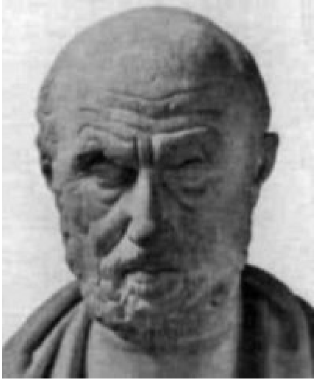
Anaxagoras

- was a great philosopher;
- was the first to think about the origins of the solar system from a non-religious viewpoint;
- first worked on the problem of squaring the circle.

Doubling the Cube

- Find the side of a cube whose volume is double that of another.
- Impossible since $\sqrt[3]{2}$ is incommensurable!

Doubling the Cube



Hippocrates of Chios

470 BC-410 BC

Doubling the Cube

Hippocrates

- was an excellent geometer;
- invented the lune in an effort to double the cube;
- is credited with proof by contradiction and writing the first Elements of Geometry.

Trisecting an Angle

- Split any given angle into three equal angles using only a compass and straightedge.
- Impossible since $\cos(20^\circ) = \cos(\pi/9)$ is incommensurable!

The Difficulties

Incommensurables contradicted Pythagorean philosophies and seemed to have punched a hole in the mathematical works of the Greeks.

What to do?

Outline

A Theorem

Theorem

The areas of two triangles having the same altitudes are to one another as their bases.

Proof (Pythagorean version)

Let the triangles be ABC and ADE with bases BC and DE lying on MN.

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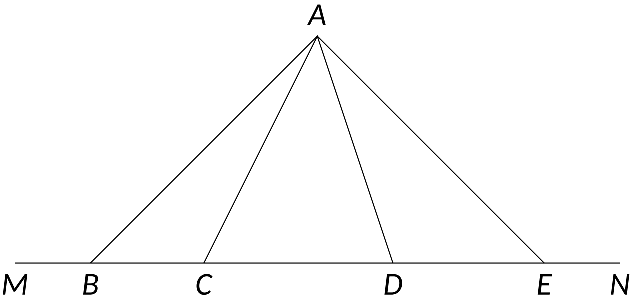
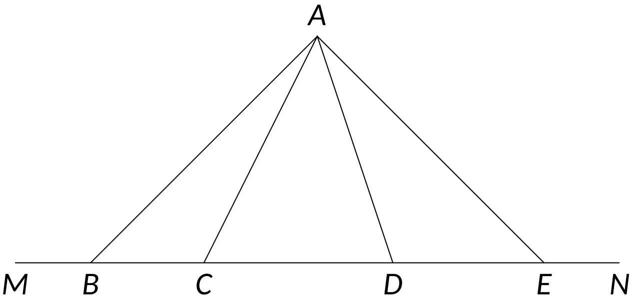


To prove: $\triangle ABC : \triangle ADE = BC : DE$.

Proof (Pythagorean version)

Let *BC* and *DE* have a common unit of measure. Let this common unit be contained *p* times in *BC* and *q* times in *DE*.



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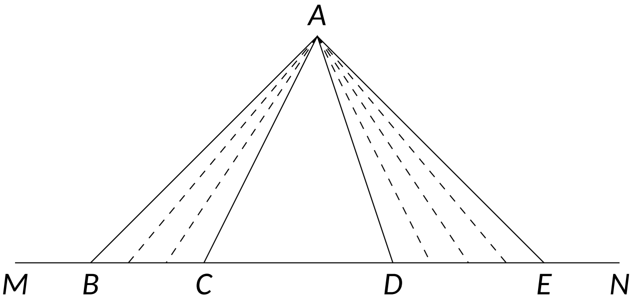
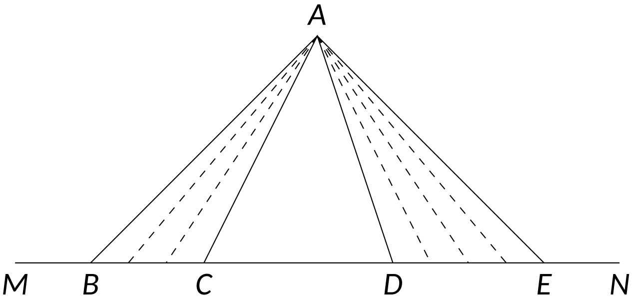


Mark off these points of measure on *BC* and *DE* and connect them with *A*.

Proof (Pythagorean version)

Then ABC and ADE are divided into p and q smaller triangles, all having a common altitude and equal bases, and thus the same area. Hence,
$\triangle ABC : \triangle ADE = p : q = BC : DE$.



The Problem with the Proof

- It is assumed that *BC* and *DE* are commensurable, when they may not be!
- Result: a flawed proof, and must be rejected
- How to fix it?

Outline

Definition of Proportion

Definition

Magnitudes are said to be in the same ratio, the first to the second and third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever be taken of the second and fourth, the former equimultiples alike exceed, are alike equal to, or are alike less than the latter equimultiples taken in corresponding order.

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Definition

Given four magnitudes A, B, C, and D, where A and B are the same kind and C and D are the same kind. Let m and n be positive integers. Then $A : B = C : D$ if

$$mA \stackrel{\geq}{<} nB \quad \text{according as} \quad mC \stackrel{\geq}{<} nD.$$

Eudoxus

- *Eudoxus* (408 BC-355 BC) developed this theory of proportion
- Viewed as a major breakthrough
- Developed the “method of exhaustion”
- Possibly developed first deductive organization of mathematics

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- Developed the "method of exhaustion"
- Possibly developed first deductive organization of mathematics
- Proved:
  - Areas of circles are to one another as the squares of their radii
  - Volumes of spheres are to one another as the cubes of their radii
  - Volume of a pyramid is one-third the volume of a prism with the same base and altitude
  - Volume of a cone is one-third that of the corresponding cylinder

A Theorem

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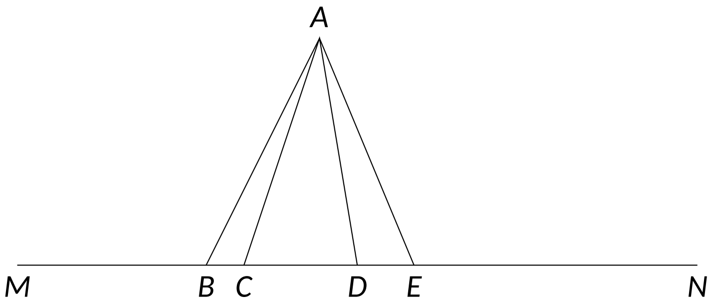
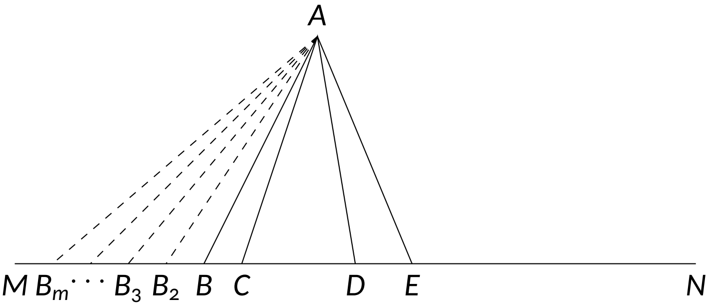
The areas of two triangles having the same altitudes are to one another as their bases.



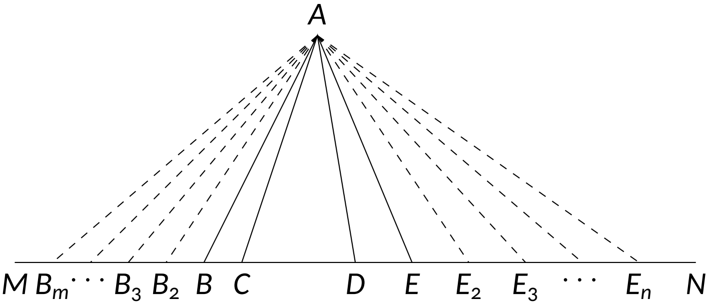
Proof (Eudoxian version)

Mark off $m - 1$ segments equal to CB and connect the points of division with A.



Proof (Eudoxian version)

Next, mark off $n-1$ segments equal to DE and connect the points of division with A.



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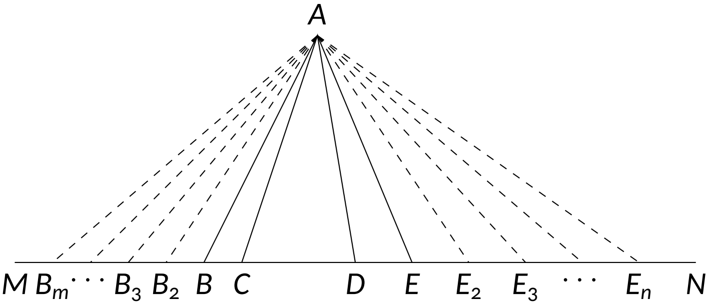

Now, $B_mC = m \cdot BC$ and $\triangle AB_mC = m \cdot \triangle ABC$. Also, $DE_n = n \cdot DE$ and $\triangle ADE_n = n \cdot \triangle ADE$.

Proof (Eudoxian version)

Now, by earlier results,

$$\triangle AB_mC \underset{<}{\overset{\geq}{=}} \triangle ADE_n \quad \text{according as} \quad B_mC \underset{<}{\overset{\geq}{=}} DE_n.$$

Proof (Eudoxian version)

Now, by earlier results,

$$\triangle AB_mC \stackrel{\geq}{<} \triangle ADE_n \quad \text{according as} \quad B_mC \stackrel{\geq}{<} DE_n.$$

Hence,

$$m \cdot \triangle ABC \stackrel{\geq}{<} n \cdot \triangle ADE \quad \text{according as} \quad m \cdot BC \stackrel{\geq}{<} n \cdot DE.$$

Proof (Eudoxian version)

Now, by earlier results,

$$\triangle AB_mC \gtreqless \triangle ADE_n \quad \text{according as} \quad B_mC \gtreqless DE_n.$$

Hence,

$$m \cdot \triangle ABC \gtreqless n \cdot \triangle ADE \quad \text{according as} \quad m \cdot BC \gtreqless n \cdot DE.$$

Thus, by the Eudoxian definition of proportion,

$$\triangle ABC : \triangle ADE = BC : DE.$$

The Advantages of the Proof

- It is *nowhere* assumed that BC and DE are commensurable!
- Result: a proof that applies to commensurable and incommensurable cases

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- Result: a proof that applies to commensurable and incommensurable cases
- Precursor to the idea of a limit

Outline

Homework

- How did Ancient cultures write fractions anyway?
  Math Through the Ages, Sketch 4

Next: Order From Chaos