Last-Minute Problems, No. 1

$\boxed{1}$ TEXTBOOK PROBLEM. [2] p.73 #2.

$\boxed{2}$ WRITTEN NUMBERS. [3] Write 574 and 475 in **2a)** Egyptian hieroglyphics; **2b)** Greek alphabetic numerals; and **2c)** Babylonian cuneiform.

$\boxed{3}$ BABYLONIAN SQUARE ROOTS. [9] The Babylonian approximation of $\sqrt{a^2+h}$ is given by

$$a+\frac{h}{2a}.$$

3a) Take $a=4/3$ and $h=2/9$ to approximate $\sqrt{2}$. (This is an approximation used by the Babylonians.)

3b) Take $a=17/12$ and $h=-1/144$ to approximate $\sqrt{2}$.

3c) Take $a=2$ and $h=1$ to approximate $\sqrt{5}$.

3d) This approximation was still used in the 20th century as a quick estimate for square roots. Usually, a^2 would be the largest perfect square less than a^2+h where $0<h<a^2$. For example, to estimate $\sqrt{70}$, we would take $a^2=64$, which implies $h=70-64=6$. Then

$$\sqrt{70}=\sqrt{8^2+6}\approx 8+\frac{6}{2\cdot 8}=8\frac{3}{8}=\frac{67}{8}.$$

Interestingly, 67/8 as a decimal is 8.375, and my TI-84 tells me that $\sqrt{70}\approx 8.367$. So this method of approximation is good enough for a rough estimate. Determine similar estimates using this "Babylonian" method for $\sqrt{56}$, $\sqrt{90}$, and $\sqrt{2519}$.

3e) A better approximation formula (used only once by the Babylonians) is

$$\sqrt{a^2+h}\approx a+\frac{h}{2a}-\frac{h^2}{8a^3}.$$

Repeat parts 3a, 3c, and 3d using this better approximation.

$\boxed{4}$ IT'S ALL GREEK TO ME. [3] In the alphabetic Greek numeral system the numbers 1000, 2000, ..., 9000 were often represented by a mark to the left of the symbols for 1, 2, ..., 9. Thus, whereas α is the symbol for 1, $/\alpha$ is 1000. The number 10,000, or *myriad*, was denoted by M, and the number of myriads written above. Thus 20,000 (2 myriads) and 300,000 (30 myriads) appeared as M^β, and M^λ. The number 4,000,000 (400 myriads) appeared as M^υ. For example, here's how the number 345,720 would be written. Since

$$\begin{aligned} 340,000 &\text{ is } M^{\lambda\delta}, \\ 5,000 &\text{ is } /\eta, \\ 700 &\text{ is } \psi, \text{ and} \\ 20 &\text{ is } \kappa, \end{aligned}$$

then 345,720 is $M^{\lambda\delta}/\eta\psi\kappa$.

4a) Write, in alphabetic Greek, the numbers 5,780, 72,803, 450,082, and 3,257,888.

4b) In order to write numbers less than 1000, how many different symbols must one memorize in the alphabetic Greek numeral system?

$\boxed{5}$ False Position Must Be True. [3] Solve the following problems, both from the *Rhind Papyrus*, by the method of false position. Do not use modern algebra.

5a) A quantity and its 1/7 added together become 19. What is the quantity?

5b) A quantity, its 2/3, its 1/2, and its 1/7 added together become 33. What is the quantity?

$\boxed{6}$ Then Double False Position Must Be False. [5] One of the oldest methods for approximating the real roots of an equation is the method of *double false position*. This method seems to have originated in India and was used by the Islamic mathematicians; however, there is evidence that the method was used in China first. In brief, the method (in modern notation) is this: Let x_1 and x_2 be two numbers lying close to and on each side of a root x of the equation $f(x) = 0$. Then the intersection with the x-axis of the line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ gives an approximation x_3 to the desired root. Symbolically,

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}.$$

The process can now be repeated with the appropriate pair x_1, x_3 or x_2, x_3.

For example, to solve $f(x) = x^2 - 3x - 15 = 0$, we first pick two numbers which make $f(x)$ as close to zero as possible, where one number gives a positive value and the other gives a negative value. We pick $x = 5$ and $x = 6$ since $f(5) = -5$ and $f(6) = 3$. Next, we find the line between the two points $(5, -5)$ and $(6, 3)$ to be $y = 8x - 45$. This line intersects the x-axis at $x = 45/8 = 5.625$. Notice that the "formula" above gives this number also, where we use $(5, -5)$ as $(x_1, f(x_1))$ and $(6, 3)$ as $(x_2, f(x_2))$:

$$\frac{6f(5) - 5f(6)}{f(5) - f(6)} = \frac{6 \cdot -5 - 5 \cdot 3}{-5 - 3} = \frac{-45}{-8} = 5.625.$$

This number, $x = 5.625$, is our first approximation to the root of $x^2 - 3x - 15$. Then, since $f(5.625) = -0.234375$, which is negative, we can repeat this process with $f(5.625)$ and $f(6)$ (since $f(6)$ is positive).

6a) Use double false position to approximate the root of $x^3 - 36x + 72 = 0$ which lies between 2 and 3 to two decimal places.

6b) Use double false position to approximate the root of $x - \tan x = 0$ which lies between 4.4 and 4.5 to two decimal places.

$\boxed{7}$ Egyptian Pi. [2] In the *Moscow Papyrus*, the area of a circle is repeatedly taken as equal to the area of a square whose side length is 8/9 of the diameter of the circle. This implies that the Egyptians used what value for π?

8 A Mysterious Number. [2] Found in an Egyptian papyrus, written in Greek, is the number below, expressed as a sum of fractions.

$$12 + \frac{2}{3} + \frac{1}{15} + \frac{1}{24} + \frac{1}{32}$$

What is this number as a modern decimal? Take on the role of an historian: for what value or operation is this number an approximation?

9 Babylonians Got 𒌋 𒌍𒐝 Problems But Base 60 Ain't 𒁹. [7] Solve, using your modern knowledge and formauls, the following problems found on a Babylonian clay tablet dating from 1800 BC. The numerical values in the problem are in Babylonian base-60, where the commas separate powers of 60 and a semicolon acts as the "decimal" point (actually, it would be the *sexagesimal* point). For example, $2, 1; 50$ is translated as

$$2, 1; 50 = 2 \cdot 60 + 1 + \frac{50}{60} = 121\frac{5}{6}.$$

9a) An area consisting of the sum of two squares is $16, 40$. The side of one square is 10 less than $0; 40$ of the side of the other square. What are the sides of the squares? (Your answer does not need to be in Babylonian numerals!)

9b) One leg of a right triangle is 50. A line parallel to the other leg and at a distance 20 from that leg cuts off a right trapezoid of area $5, 20$. Find the lengths of the bases of the trapezoid.

9c) Find the area of a trapezoid with bases 14 and 50 and sides 30. (Notice that the numerals used here are Babylonian, but does that really matter?)

10 Egyptians Got ∩∩∩∩∩∩∩∩∩||||||||| Problems But Hieroglyphics Ain't |. [3] Solve these problems which are taken from the *Moscow Papyrus* without permission. (Hopefully some ancient Egyptian mummy won't come after me for stealing math problems.)

10a) The area of a rectangle is 12 and the width is $3/4$ the length. What are the dimensions?

10b) One leg of a right triangle is 2 and $1/2$ times the other and the area is 20. What are the dimensions?

11 The Seqt of a Pyramid. [2] The Egyptians measured the steepness of a face of a pyramid by the ratio of the "run" to the "rise"—that is, by giving the reciprocal of what we consider the slope of the face of the pyramid. The vertical unit (the "rise") was the *cubit* and the horizontal unit (the "run") was the *hand*; there were 7 hands in a cubit. (So that means the "rise" and "run" are different units.) With these units, the measure of steepness was called the *seqt* of a pyramid.

11a) Solve Problem 56 of the *Rhind Papyrus* which asks: What is the seqt of a pyramid 250 cubits high and with a square base 360 cubits on a side? (Don't forget to convert the horizontal distance to hands.)

11b) The great pyramid of Cheops has a square base 440 cubits on a side and a height of 280 cubits. What is the seqt of this pyramid?

12 BABYLONIAN QUADRATICS. [9] A problem from Babylon asks for the side of a square if the area of the square diminished by the side of the square is 870 (from tablet BM 13901). A literal translation of the problem, by Babylonian scholar Eleanor Robson, goes like this:

> I took away my square-side from inside the area and it was $14, 30$. You put down 1, the projection. You break off half of 1. You combine $0; 30$ and $0; 30$. You add $0; 15$ to $14, 30$. $14, 30; 15$ squares $29; 30$. You add $0; 30$ which you combined to $29; 30$ so that the square-side is 30.

This problem is clearly geometric in approach: you are taking away, projecting, breaking off, squaring, and combining. However, if we modernize the language and use our numerals, we get the following approach which is more algebraic in nature.

> Take half of 1, which is 1/2; multiply 1/2 by 1/2, which is 1/4; add the 1/4 to 870, to obtain 870 and 1/4. This last is the square of 29 and 1/2. Now add 1/2 to 29 and 1/2; the result is 30, which is the side of the square.

If we insist on overlaying our modern algebraic approach, then the problem reduces to solving the equation $x^2 - px = q$, where the side of the square is x. Then the area of the square is x^2, and we take away a side, which has length x; hence, $p = 1$ in this problem. The square minus the side should be 870; hence $q = 870$ in this problem.

12a) Show that the Babylonian solution above is exactly equivalent to solving the quadratic equation $x^2 - px = q$ by the formula

$$x = \sqrt{\left(\frac{p}{2}\right)^2 + q} + \frac{p}{2}.$$

(Demonstrate this by using the quadratic formula on $x^2 - px - q = 0$, and rewriting the result.)

Another Babylonian tablet solves the equation $11x^2 + 7x = 25/4$ by first multiplying through by 11 to obtain $(11x)^2 + 7(11x) = 625/4$, which, by setting $y = 11x$, has the "normal" form $y^2 + py = q$. This is solved by the formula

$$y = \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2};$$

and, finally, $x = y/11$.

12b) Solve $7x^2 + 5x = 2$ by this method, obtaining the only positive solution.

12c) What is the negative solution?

Other kinds of problems which result in quadratic equations appeared in various Babylonian tablets. Solve these.

12d) From tablet BM 13901: The sum of the areas of two squares is 1525. The side of the second square is 2/3 of that of the first plus 5. Find the sides of each square.

12e) Another tablet has a problem that leads to the system of equations

$$x + y = 5\frac{5}{6}, \qquad \frac{x}{7} + \frac{y}{7} + \frac{xy}{7} = 2.$$

Solve this by first multiplying the second equation by 7 and then subtracting the first equation from the second. Then the system is easier to manage; solve it.