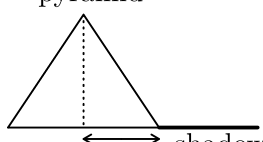
Last-Minute Problems, No. 2

1 TEXTBOOK PROBLEMS. [3.5] p.127 #1 and #3; p.128 #7.

2 THE THALES PUZZLE. [3.5] There are two versions of how Thales, when in Egypt, evoked admiration by calculating the height of a pyramid by shadows. The earlier account given by Hieronymous, a pupil of Aristotle, says that Thales determined the height of the pyramid by measuring the shadow it cast at the moment a man's shadow was equal to his height. The later version, given by Plutarch, says that he set up a stick and then made use of similar triangles. Both versions fail to mention the very real difficulty, in either case, of obtaining the length of the shadow of the pyramid—that is, the distance from the shadow of the top of the pyramid *to the center of the base of the pyramid*.



This unaccounted-for difficulty is called the Thales puzzle: Devise a method, based on shadow observations and similar triangles and independent of latitude and specific time of day or year, for determining the height of the pyramid. (*Hint:* Use two shadow observations spaced a few hours apart.)

3 CIRCLES AND SAGITTAS AND CHORDS, OH MY. [3.5] A *sagitta* of a circle is a line segment from the midpoint of a chord of a circle to the circle and is also perpendicular to the chord.

3a) Interpret mathematically the following, found on a Babylonian tablet dating from 2600 BC.

> 60 is the circumference, 2 is the sagitta, find the chord.
> You double 2 and get 4, do you not see? Take 4 from 20, you get 16. Square 20, you get 400. Square 16, you get 256. Take 256 from 400, you get 144. Find the square root of 144. 12, the square root, is the chord. Such is the procedure.

3b) Using part 3a as a guide, derive a correct formula for the length of a chord in terms of the radius r and sagitta s.

4 TRUNCATED PYRAMIDS. [2.5] The Egyptians discovered the correct formula for the volume of a truncated square pyramid:

$$V = \frac{h}{3}(a^2 + ab + b^2),$$

where a is the length of the lower base, b is the length of the upper base, and h is the height. The Chinese also discovered the correct formula

$$V = \frac{h}{6}[(2a + b)a + (2b + a)b].$$

The Babylonians, although they had a correct formula, used the following *incorrect* formula more often:

$$V = \frac{h}{2}(a^2 + b^2).$$

Consider the truncated pyramid of height 6, base 4, and top 2, as found in a problem of the *Moscow Papyrus*. Calculate the correct volume using either the Egyptian or Chinese formulas, then calculate the incorrect volume using the Babylonian formula. What is the percentage error* given by the Babylonian formula? Do you think the Babylonians could "tell" they were using the incorrect formula?

5 EMPIRICAL MATHEMATICS. [3.5] The idea of averaging is important in empirical work. Thus, we find in the *Rhind Papyrus* the area K of a quadrilateral having successive sides a, b, c, and d given by

$$K = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right).$$

5a) How does this formula demonstrate averaging? (Don't overthink this.)

5b) This formula gives too large a result for all nonrectangular quadrilaterals. For instance, consider a parallelogram with sides 20, 5, 20, and 5 and a height of 4 between the sides of length 20. The area is known to be (base) $\times$ (height) $=$ $4 \times 20 = 80$. Calculate the incorrect area given by the Egyptian formula, and determine the percentage error.

5c) Now assume that the Egyptian formula is correct. Under this assumption, show that the area of a triangle would be given by half the sum of two sides multiplied by half the third side.†

5d) What does the word "empirical" mean? (Look it up!) How does it apply here?

6 THE MUSIC OF PYTHAGORAS. [3.5] The interval from the first note of the scale to the second is also called a *tone*, and the seven intervals in the usual scale C, D, E, F, G, A, B, C, are tone, tone, semitone, tone, tone, tone, semitone. (A semitone is half a tone.) Thus the interval from C to the next C is six tones, and should correspond to the frequency ratio of 2.

6a) If a tone corresponds to a frequency ratio of 9/8 (as the Pythagoreans thought), then why does an interval of six Pythagorean tones correspond to $9^6/8^6$?

6b) Divide $9^6/8^6$ by 2 and write the result as a fraction, thereby proving that $9^6/8^6$ is not 2.

6c) In modern music, the interval from C to the C one octave higher is divided into 12 equal semitones with the help of extra notes called "sharps." These are C#, D#, F#, G#, and A#, with the sharp note coming after the tone of the same letter. (These are the black keys on a piano.) Which note divides the

*Percentage error is |estimated value − actual value| $\times$ 100 $\div$ actual value.

†This incorrect formula for the area of a triangle is found in a property deed from Egypt, dating 1500 years after the *Rhind Papyrus*.

octave into two equal intervals? Three equal intervals? Four equal intervals? Six equal intervals?

7 MAYBE IT SHOULD BE THE "BABYLONIAN THEOREM". [6] Each Pythagorean triple (a, b, c) in the Babylonian tablet *Plimpton 322* can be "explained" in terms of a simpler number x, given in the following table. (The numbers x are *not* in *Plimpton 322* but they provide a plausible explanation for it, as we will see.)

row	a	b	c	x
1	120	119	169	12/5
2	3456	3367	4825	64/27
3	4800	4601	6649	75/32
4	13500	12709	18541	125/54
5	72	65	97	9/4
6	360	319	481	20/9
7	2700	2291	3541	54/25
8	960	799	1249	32/15
9	600	481	769	25/12
10	6480	4961	8161	81/40
11	60	45	75	2
12	2400	1679	2929	48/25
13	240	161	289	15/8
14	2700	1771	3229	50/27
15	90	56	106	9/5

For each line in the table, $\dfrac{b}{a} = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right)$ and $\dfrac{c}{a} = \dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$.

7a) Check that when $x = 12/5$, we get $119/120$ and $169/120$.

7b) Check the next three rows in the table.

7c) Is the 3-4-5 right triangle in the table? How do you know?

7d) Verify by working out the algebra that

$$\left(\frac{1}{2}\left(x - \frac{1}{x}\right)\right)^2 + 1 = \left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right)^2.$$

7e) The numbers x are "simple" in the sense that they are built from the numbers 2, 3, and 5. For example, $12/5 = (2^2 \cdot 3)/5$ and $125/54 = 5^3/(2 \cdot 3^3)$. Numbers divisible by 2, 3, and 5 result in terminating base-60 "decimals" so these fractions are simple to the Babylonians. Write the numbers x in rows two and three in terms of 2, 3, and 5.

8 THE PYTHAGOREAN THEOREM OVER AND OVER AND OVER AND . . . . [13.5] Certainly no mathematical proposition boasts a greater number of different proofs than the theorem of Pythagoras, for which well over 400 different arguments can be found in E.S. Loomis' *The Pythagorean Proposition*, published in 1968. These may seem to approach mathematical overkill; on the other hand, they should satisfy even the most hard-boiled sceptic.

For each proof outlined below, we begin with a right triangle BAC with right angle at A (thus the hypotenuse is labeled a and the legs are b and c). As you work through the details of these proofs, ask yourself where we use the assumption that we have a *right* triangle.

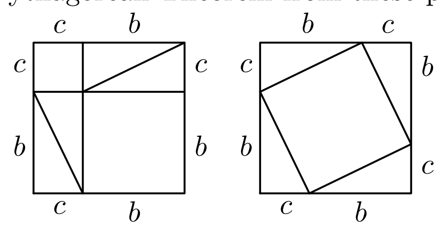
8a) This is the proof that we may attribute to Pythagoras himself. We begin with identical squares of side length $b+c$, decomposed as shown to the right.

a1: Show that the area of the left-hand square is $2bc + b^2 + c^2$.

a2: Prove that the inner figure on the right is itself a square.

a3: Show that the area of the right-hand side square is $2bc + a^2$.

a4: Now prove the Pythagorean Theorem from these preliminaries.

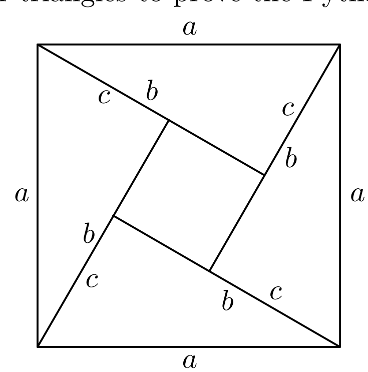


8b) This proof is due to the 12th century Indian mathematician Bhaskara. Assemble four copies of the original triangle as shown below.

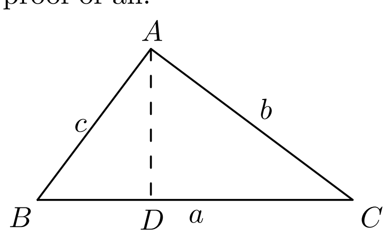
b1: Prove that the large quadrilateral is a square.

b2: Prove that the inner quadrilateral is a square.

b3: Equate the area of the large square with the total areas of the small square and the four triangles to prove the Pythagorean Theorem.



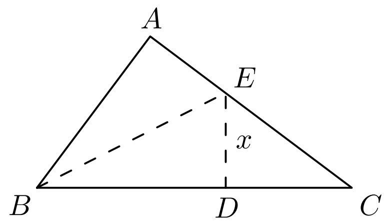
8c) This proof is usually attributed to the 17th century British mathematician John Wallis, although it surely had been discovered prior to him. It is regarded as the shortest proof of all.



c1: From A draw altitude $\overline{AD}$ to the hypotenuse and prove $\triangle ADC \sim \triangle BAC \sim \triangle BDA$.

c2: Conclude that $b/CD = a/b$ and that $c/DB = a/c$. From this, complete the proof.

8d) Here is another similarity proof. In right triangle BAC, mark off $BD = BA$, then bisect $\angle ABC$ by line BE, where E is on side AC. Also, draw ED and call its length x.

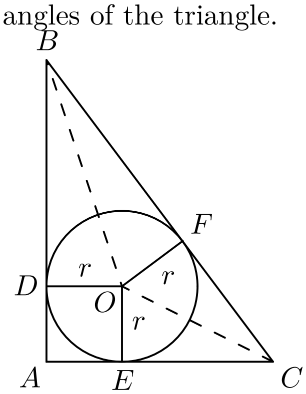


d1: Prove $\triangle BAE \cong \triangle BDE$.

d2: Show $\triangle EDC \sim \triangle BAC$.

d3: Set up the resulting proportions from part d2 and use these to eliminate x and thereby derive the theorem.

8e) Here is a clever "inscribed circle" proof. Again begin with $\triangle BAC$. Inscribe within a circle having center O and radius r. Draw OD, OE, and OF, as shown. You may use one key fact from Euclid about inscribed circles: (Prop. IV.4) The point O – the center of the inscribed circle – is the intersection of the three bisectors of the angles of the triangle.



e1: Carefully prove that $\triangle BOD \cong \triangle BOF$ and $\triangle COE \cong \triangle COF$.

e2: Prove quadrilateral $AEOD$ is a square.

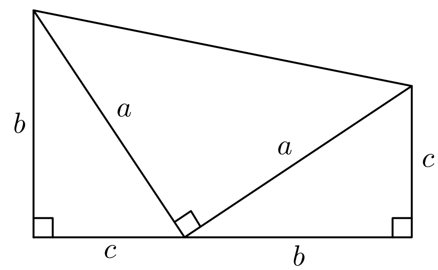
e3: Explain why $b + c - 2r = a$ and solve this for r.

e4: Why is the area of $\triangle BAC$ equal to $bc/2$?

e5: Decomposing $\triangle BAC$ into the square and the two pairs of congruent triangles, prove that the area of $\triangle BAC$ is equal to $r(b + c - r)$.

e6: Finally, equate the two expressions for the area of $\triangle BAC$ from parts e4 and e5, substitute for r from part e3, and do some more algebra to derive the theorem.

8f) The last proof is due to Congressman (later President) James A. Garfield of Ohio, who published it in the *New England Journal of Education* in 1876. This proof depends on calculating the area of the trapezoid in the figure below in two different ways: by the formula for the area of the trapezoid and by the sum of the areas of the triangles. Carry out this proof in detail.



9 Two From Nine. [3] The most famous ancient Chinese mathematical work is the *Jiuzhang suanshu*, or *Nine Chapters on the Mathematical Art*. Compiled around 200 BC by one of ancient China's most renowned mathematicians, Liu Hui, but probably written 900 years earlier, this work formed the basis of mathematics education in China for nearly a millennium. Solve the following two problems from the *Nine Chapters*.

9a) The height of a wall is 10 *ch'ih*.[‡] A pole of unknown length leans against the wall so that its top is even with the top of the wall. If the bottom of the pole is moved 1 *ch'ih* farther from the wall, the pole will fall to the ground. What is the length of the pole?

9b) A square pond has side 10 *ch'ih* with a reed growing in the center whose top is 1 *ch'ih* out of the water.[§] If the reed is pulled to the shore, the top just reaches the shore. What is the depth of the water and the length of the reed?

10 Another Generalization. [3] The Islamic mathematician Thābit ibn Qurra was a great scholar of the ninth century AD. He was a teacher at the House of Wisdom in Baghdad, the center of the world's learning during this period (much like Alexandria was the center of learning during the Greek times). He discovered the following marvelous theorem:

> Given any triangle ABC with B' and C' points on BC such that $\angle AB'B = \angle AC'C = \angle A$, then $(AB)^2 + (AC)^2 = BC(BB' + CC')$.

Prove that if $\angle A$ is a right angle, then this theorem becomes the Pythagorean Theorem.

11 Pythagorean Triples Are Never Identical. [4.5] An ordered triple (a, b, c)—called a Pythagorean Triple—is an ordered triple of numbers that are also sides of a right triangle. For example, $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples; in fact, they are called *primitive* Pythagorean triples since the greatest common factor of the three numbers is 1. For instance, $(6, 8, 10)$ is a Pythagorean triple, but it is not primitive since the greatest common factor of 6, 8, and 10 is 2.

11a) Using algebra, show that, for any positive integer n, the three numbers $2n$, $n^2 - 1$, and $n^2 + 1$ constitute a Pythagorean triple. It is believed that the Pythagoreans knew this.

[‡] The *ch'ih* is an ancient Chinese unit of length, which is approximately 13.1 inches.
[§] A reed grows from the bottom of a pond.

11b) Let integers u and v have greatest common factor 1, let one be even and one odd, and let $u > v$. Then every primitive Pythagorean triple (a, b, c) is given by

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2.$$

For example, if $u = 2$ and $v = 1$, then $a = 3$, $b = 4$, and $c = 5$. Find the 16 primitive Pythagorean triples for which $c < 100$. *Suggestion:* Make a table with column headings u, v, a, b, and c to organize your work.

11c) Prove that no isosceles right triangle exists whose sides are integers.