Last-Minute Problems, No. 3

1 TEXTBOOK PROBLEMS. [7] p.93 #4, #5, #6, and #7; p.94 #8

2 GEOMETRICAL EXPERIMENTS—NO LAB EQUIPMENT NEEDED. [6] For the following problems, solve the problems by experiment and intuition. (There are no wrong answers—except an unjustified one!)

2a) Let F, V, and E denote the number of faces, vertices, and edges of a polyhedron. For the tetrahedron, cube, triangular prism, pentagonal prism, square pyramid, pentagonal pyramid, cube with one corner cut off, and cube with a square pyramid erected on one face, we find

$$V - E + F = 2.$$

Do you feel that this formula holds for all polyhedra? Why?

2b) There are convex polyhedra all faces of which are triangles (i.e., a tetrahedron), all faces of which are quadrilaterals (i.e., a cube), all faces of which are pentagons (i.e., a regular dodecahedron). Do you think the list can be continued? Why?

2c) Consider an ellipse with semiaxes a and b. If $a = b$ the ellipse becomes a circle and the two expressions

$$P_1 = \pi(a+b) \quad \text{and} \quad P_2 = 2\pi\sqrt{ab}$$

each become $2\pi a$, which gives the perimeter (circumference) of the circle. This suggests that P_1 or P_2 may give the perimeter of any ellipse. Which one is the right one? Why?

2d) If the inside of a race track is a noncircular ellipse, and the track is of constant width, is the outside of the track also an ellipse? How could you justify your answer? (Notice I am not asking you to prove your answer is the right one, but simply asking you to think about how you go about justifying it.)

2e) Two ladders, 60 feet long and 40 feet long, lean from opposite sides across an alley lying between two buildings, the feet of the ladders resting against the bases of the buildings. If the ladders cross each other at a distance of 10 feet above the alley, how wide is the alley?

Find an approximate solution from drawings. Then find the solution by solving the following equation in your calculator: If a and b are the lengths of the ladders, c the height at which they cross, and x the width of the alley, then x satisfies

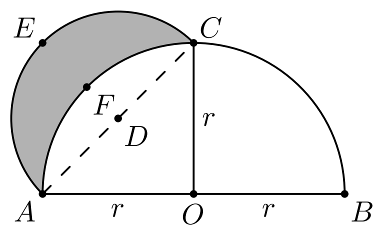
$$\frac{1}{\sqrt{a^2 - x^2}} + \frac{1}{\sqrt{b^2 - x^2}} = \frac{1}{c}.$$

3 IRRATIONALS. [3]

3a) Prove that the straight line through the points $(0,0)$ and $(1,\sqrt{2})$ passes through no point, other than $(0,0)$, with integer coordinates.

3b) Prove that $\log_{10} 2$ is irrational.

4 THE QUADRATURE OF THE LUNE. [8] Hippocrates of Chios was an ancient Greek mathematician and astronomer who lived in the 5th century BC.¶ He is credited with writing the first book on geometry. Below is the figure Hippocrates used in his geometric proof of the area of the lune. The lune is the shaded area in the figure. In what follows, you will prove his result algebraically (which is, of course, totally alien to Greek mathematics).



4a) Letting r be the radius of semicircle ACB, find the area of triangle AOC in terms of r.

4b) Now determine the area of circular segment $AFCD$ in terms of r. (Use the post-Hippocratean fact that the area of a circle is πr^2.)

4c) Next, find the area of semicircle $AECD$ in terms of r.

4d) Using parts 4b and 4c, find the area of lune $AECF$ in terms of r.

4e) Finally, put all of this together to explain why Hippocrates' lune is quadrable. (That is, why is it possible to construct a square whose area is equal to that of the lune?)

5 THE ANCIENT GREEKS KNEW THE EARTH WAS ROUND, SO GET OVER IT, FLAT-EARTHERS. [3.5] The Greek philosopher Eratosthenes (276-194 BC), in 240 BC, made a measurement of the earth. He observed at Syene, at noon on the summer solstice, that a vertical stick cast no shadow, while at Alexandria (which he believed to be on the same meridian as Syene) the sun's rays were inclined to the vertical 1/50 of a complete circle. He then calculated the circumference of the earth from the known distance of 5000 stades between Alexandria and Syene. Obtain Eratosthenes' result of 250,000 stades for the circumference of the earth. There is reason to believe that a stade was equal to about 559 feet. Assuming this, calculate from the above result the diameter of the earth in miles.‖

6 THE USEFULNESS OF DEFINITIONS. [3.5] The definition of a technical term (beyond primitive ones) of a logical discourse serves merely as an abbreviation for a complex combination of terms already present. Thus a new term introduced by definition is really arbitrary, though convenient, and may be entirely ignored; but then the discourse would immediately increase in complexity. Consider, for

¶Not to be confused with the other Hippocrates, who lived around the same time, who is credited as the father of Western medicine, and for whom the physician's "Hippocratic Oath" is named.

‖The actual diameter of the earth to the nearest mile is 7900 miles.

example, the following definitions taken from a geometry text: The *diagonals* of a quadrilateral are the two straight line segments joining the two pairs of opposite vertices of the quadrilateral; *parallels* are straight lines lying in the same plane and never meeting, however far they are extended in either direction; a *parallelogram* is a quadrilateral having its opposite sides parallel.

6a) Without using any of the italicized words above, restate the proposition: "The diagonals of a parallelogram bisect each other."

6b) By means of appropriate definitions, reduce the the following sentence to one containing no more than five words: "The movable seats with four legs were restored to a sound state by the person who takes care of the building."

$\boxed{7}$ Time To Get Mean. [12] If a and b are two real numbers, the following means have been found useful:

α. arithmetic mean: $\dfrac{a+b}{2}$

β. geometric mean: $\sqrt{ab}$

γ. harmonic mean: $\dfrac{2ab}{a+b}$

δ. heronian mean: $\dfrac{a+\sqrt{ab}+b}{3}$

ε. contraharmonic mean: $\dfrac{a^2+b^2}{a+b}$

ζ. root-mean-square: $\sqrt{\dfrac{a^2+b^2}{2}}$

η. centroidal mean: $\dfrac{2(a^3-b^3)}{3(a^2-b^2)}$

Here is one way to explain what these means represent. Let a and b, with $a > b$, denote the lengths of the lower and upper bases of a trapezoid. Then any line segment parallel to the bases with endpoints on the other sides is some *mean* of a and b:

α. arithmetic: bisects the sides of the trapezoid

β. geometric: divides the trapezoid into two similar trapezoids

γ. harmonic: passes through the intersection of the diagonals

δ. heronian: 1/3 of the way from the arithmetic mean to the geometric mean

ε. contraharmonic: as far below the arithmetic as the harmonic is above the arithmetic

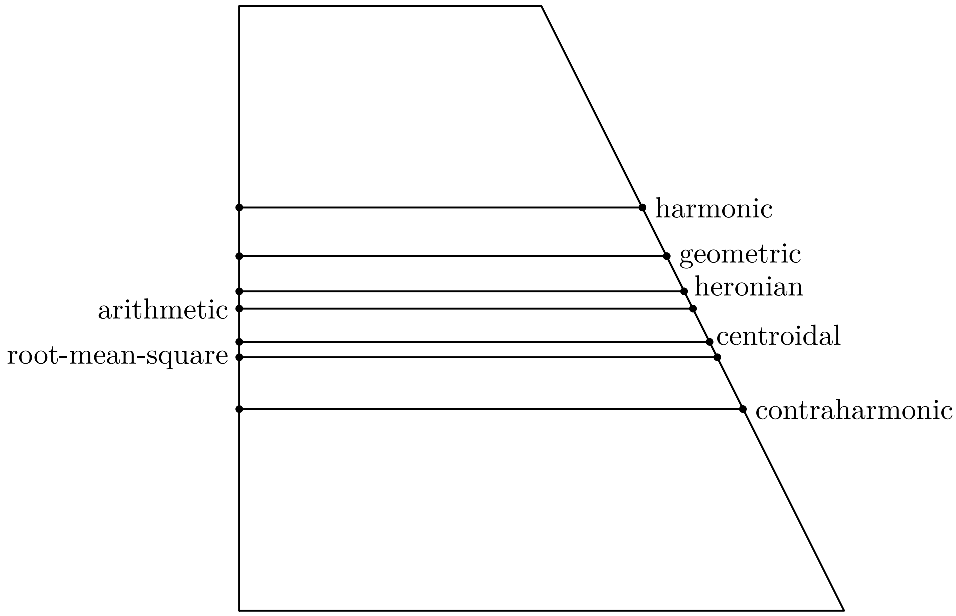
ζ. root-mean-square: bisects the area of the trapezoid

η. centroidal: passes through the center of gravity (the centroid) of the trapezoid

The figure on the next page illustrates these line segments.

7a) Find each of the seven means of 8 and 16.

7b) A trapezoid has bases of lengths 6 and 12. What is the length of the line that cuts the trapezoid into two trapezoids of equal area? What is the length of the line that cuts the trapezoid into two similar trapezoids?



7c) Let s be the side of a square inscribed in a right triangle and having one angle coinciding with the right angle of the triangle. Show that s is half the harmonic mean of the legs of the triangle.

7d) A car travels at the rate r_1 mph from A to B, and then returns at the rate r_2 mph from B to A. Show that the average rate for the round trip is the harmonic mean of r_1 and r_2.

Equally used in ancient times were the "progression" interpretations of the arithmetic, geometric, and harmonic means. We say that three numbers p, q, r (in that order) are in arithmetic progression if

$$q - p = r - q;$$

they are in geometric progression if

$$\frac{q}{p} = \frac{r}{q};$$

and they are in harmonic progression if

$$\frac{q-p}{p} = \frac{r-q}{r}.$$

7e) If a, b, and c are in harmonic progression, then prove that their reciprocals $1/c$, $1/b$, and $1/a$, are in arithmetic progression.

7f) If a^2, b^2, and c^2 are in arithmetic progression, then prove that $b+c$, $c+a$, and $a+b$ are in harmonic progression.

7g) Since 8 is the harmonic mean of 12 and 6, Philolaus, a Pythagorean of about 425 BC, called the cube a "geometrical harmony." Why a cube and not some other solid?

$\boxed{8}$ GEOMETRIC ALGEBRA. [7] Let r and s denote the roots of the equation

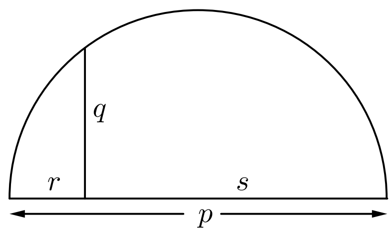
$$x^2 - px + q^2 = 0$$

where p and q are positive numbers.

8a) Show that $r + s = p$ and $rs = q^2$.

8b) Show that if $q \leq p/2$ then r and s are both positive.

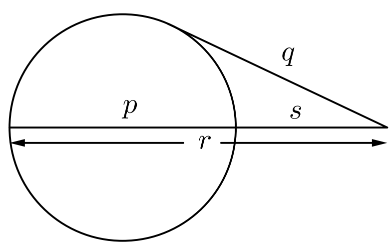
8c) To solve the equation $x^2 - px + q^2 = 0$ geometrically, we must find segments r and s from given segments p and q. The figure below is the construction made, from segments p and q, to find r and s. Explain the construction and explain why it works.



8d) Now consider the equation

$$x^2 - px - q^2 = 0$$

where p and q are positive numbers. Here, the roots are real, but one, say s, is negative and the other, r, is positive. To solve this equation geometrically, we again must find segments r and s from given segments p and q. The figure below is the construction made, from segments p and q, to find r and s. Explain why the construction works.



With these two examples of geometric algebra, it is easy to see why ancient cultures considered each equation a *completely different problem*.