Contents

1 Section 1.2, Vectors in Space

Objective. *Students will understand the purpose of the course and the classroom policies. Students will compute and interpret cross products and dot prodcuts of vectors.*

- Rationale for the course;
- overview of 1st semester;
- hand-out syllabus;
- brief outline of second semester;
- issue textbooks;
- describe grading policies.

Read Section 1.1 in text.

- Distance Formula. —[[Larson 79 and 80, Thomas 10.5 and 10.6]]—

- Dot product defined as $\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos\theta$. —[[Larson 81]]—
- Algebraic rules for the dot product (Eq. 1.11, page 4).
- Direction angles and direction cosines. —[[Thomas 10.4]]—
- Cross products: $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, and $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta$ is area of parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$. —[[Thomas 10.15]]—
- Algebraic rules for the cross product (Eq. 1.19, page 5).
- Vector triple products.

Homework for Day 1. Page 15, #1 parts a–d, #2

HOMEWORK ANSWERS. #1 See text

#2

a) 0

b) $\langle -1, 7, 4\rangle$

c) $\sqrt{6}$

d) 90°

e) $\langle -38, 14, 26\rangle$

f) 42 and -42

g) $\mathbf{0}$

h) $\cos\alpha = 1/\sqrt{6}$, $\cos\beta = -1/\sqrt{6}$, $\cos\gamma = 2/\sqrt{6}$

i) $6/\sqrt{30}$

2 Section 1.3, Linear Independence; Lines and Planes in Space

Objective. *Students will determine whether a given set of vectors is linearly dependent or linearly independent. Students will use vectors to represent properties of lines and planes in space.*

Linear combination: $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ for constants c_i and vectors $\mathbf{u}_i$.

Linearly independent: do not represent the same line; i.e., the only way the linear combination equals $\mathbf{0}$ is if each constant is zero.

Linearly dependent: lie on the same line; i.e., there are nonzero constants so that the linear combination equals $\mathbf{0}$.

Theorem 2.1. *Any four vectors in space are linearly dependent.*

Proof. If one of the vectors is the zero vector, we are done. So assume none of the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ is the zero vector. Then we have two cases: either $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly dependent, or $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are linearly independent.

In the first case, there exist nonzero constants such that $c_1\mathbf{u}_1+c_2\mathbf{u}_2+c_3\mathbf{u}_3 = \mathbf{0}$. But then $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + 0\mathbf{u}_4 = \mathbf{0}$, so the four vectors are linearly dependent.

In the second case, the three vectors are not coplanar, so they can form three edges of a parallelpiped. Then $\mathbf{u}_4$ can be represented as a linear combination in the other three vectors (akin to representing a vector in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$). Hence, $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 - \mathbf{u}_4 = \mathbf{0}$ so that the four vectors are linearly dependent. □

Every linearly independent triple of vectors can serve as a basis for the space.

If $P_1(x_1, y_1, z_1)$ is a point in the plane and $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is a normal vector, then P is in the plane when $\mathbf{n} \cdot \overrightarrow{P_1P} = 0$, or $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$, or $Ax + By + Cz + D = 0$.

Every linear equation $Ax + By + Cz + D = 0$ represents a plane with $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ as a normal vector. —[[Larson 84]]—

If $P_1(x_1, y_1, z_1)$ is a point on a line and $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a nonzero vector on the line, then P is on the line when $\mathbf{v} \times \overrightarrow{P_1P} = \mathbf{0}$. —[[Larson 83]]—

Hence, $\overrightarrow{P_1P}$ must be a scalar t times $\mathbf{v}$. This is the parametric representation of a line: $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$.

Distance from a point (x_0, y_0, z_0) to a line with parameter t is

$$\sqrt{(x_1 + at - x_0)^2 + (y_1 + bt - y_0)^2 + (z_1 + ct - z_0)^2}.$$

Distance from a point (x_0, y_0, z_0) to a plane $Ax + By + Cz + D = 0$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

Example 2.1. *Find the distance between* $(1, 2, 2)$ *and the line* $x = 1 + t,\ y = 2 - t,\ z = 3 + t.$

$$\sqrt{(1 + t - 1)^2 + (2 - t - 2)^2 + (3 + t - 2)^2} = \sqrt{3t^2 + 2t + 1}$$

Example 2.2. *Find parametric equations for the line passing through* $(1, 2, 2)$ *and is perpendicular to the line* $x = 1 + t,\ y = 2 - t,\ z = 3 + t.$

The line vector is $\mathbf{i} - \mathbf{j} + \mathbf{k}$; the normal vector is $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Thus, the line is $x = 1 + t,\ y = 2 + 2t,\ z = 2 + t.$

Example 2.3. *Find parametric equations for the line passing through* $(1, 2, 2)$ *and is perpendicular to the line* $x = 1+t,\ y = 2-t,\ z = 3+t$ *and perpendicular to* $x = 2 + t,\ y = 5 + 2t,\ z = 7 + 4t.$

To find the normal vector, compute the cross product of the two line vectors to get $-6\mathbf{i}-3\mathbf{j}+3\mathbf{k}$; hence, the line is $x = 1-6t,\ y = 2-3t,\ z = 2+3t.$

HOMEWORK FOR DAY 2. Page 16, #5, #6, parts a, b, and c, #7, parts a and b

HOMEWORK ANSWERS. #5 All are dependent.
#6

a) $P_1 = (2, 1, 0)$ and $\overrightarrow{P_1P} = \langle 1, 1, 5\rangle$, so the line is $x = 2+t,\ y = 1+t,\ z = 5t$.

b) Line is $x = 1 - 5t,\ y = 1 + 2t,\ z = 2 + 3t$.

c) Line is $x = 5t,\ y = -t,\ z = t$.

#7

a) z-coordinate doesn't matter; plane is $2x - y = 0$.

b) $P_1 = (1, 2, 2)$ and $\mathbf{n} = \langle -1, 5, -4\rangle$, so the plane is $-(x - 1) + 5(y - 2) - 4(z - 2) = 0$ or $-x + 5y - 4z - 1 = 0$.

3 Section 1.4, Determinants

Objective. *Students will use properties of determinants to evaluate determinants.*

A 2-by-2 determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Higher-order determinants can be reduced recursively.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Six rules of determinants.

1. Rows and columns can be interchanged.
2. Interchanging two rows(columns) changes the sign of the det.
3. A common factor of any row(column) can be placed in front of the det.
4. If one row(column) is a multiple of another, then det equals zero.
5. Determinants differing in only one row(column) can be added by adding the rows(columns) and leaving the others unchanged.
6. Row operations leave det unchanged.

Let (x_1, y_1) and (x_2, y_2) be two points in the xy-plane. Then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

is a line. If (x_3, y_3) is a third point, then

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

is the area of the triangle formed by the three points.

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three vectors in space, then one can get a determinant

$$D = \begin{vmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{vmatrix} = \mathbf{u} \cdot \mathbf{v} \times \mathbf{w} = \mathbf{w} \cdot \mathbf{u} \times \mathbf{v} = \mathbf{v} \cdot \mathbf{w} \times \mathbf{u}.$$

Called scalar triple products, $\cdot$ and $\times$ can be interchanged; and interchanging two vectors changes the sign of the product.

D is also the volume of parallelepiped with edges $\mathbf{u}, \mathbf{v}, \mathbf{w}$; $D = |\mathbf{u} \times \mathbf{v}||\mathbf{w}| \cos\phi$ where $|\mathbf{w}| \cos\phi$ is the height and $|\mathbf{u} \times \mathbf{v}|$ is the area of the base. —[[THOMAS 10.19]]—

HOMEWORK FOR DAY 3. Page 16, #4, #10

HOMEWORK ANSWERS. #10

a) -17

b) -7

c) 2

d) -1

e) -36

f) $abc[bc(c-b)-ac(c-a)+ab(b-a)]$

4 Sections 1.5, 1.6, and 1.7, Systems of Linear Equations, Matrices, Matrix Addition, and Scalar Multiplication

Objective. *Students will use Cramer's Rule to solve systems of equations. Students will understand basic matrix terminology. Students will add and subtract matrices and multiply matrices and scalars.*

Cramer's Rule: $x = D_x/D, y = D_y/D, z = D_z/D$ where D is the det of the coefficient matrix, and D_x, D_y, D_z is the det of the coefficient matrix with column 1, 2, 3 replaced by the column of equalities. If $D \neq 0$ and $D_x = D_y = D_z = 0$ then there is only the trivial solution $x = y = z = 0$. If $D = D_x = D_y = D_z = 0$ then there are infinitely many solutions. This implies that the row vectors are linearly dependent. Then we can find nonzero constants; those constants times a parameter t solve the system.

Example 4.1. *Solve the system:* $\begin{cases} 2x - 3y + z &= 0 \\ x + y - z &= 0 \\ x - 4y + 2z &= -1 \end{cases}$

Example 4.2. *Solve the system:* $\begin{cases} 2x - y + 2z &= -1 \\ x - 2y + 3z &= -4 \\ 3x + 2y + 2z &= 3 \end{cases}$

Example 4.3. *Solve the system:* $\begin{cases} x + y + z &= 0 \\ -4x + 2y - z &= -3 \\ -5x + y - 2z &= -3 \end{cases}$

Example 4.4. *Solve the system:* $\begin{cases} x + y + z + w &= 0 \\ x - z &= 0 \\ 2x + y &= 0 \\ y - w &= 0 \end{cases}$

A *matrix* is a rectangular array of real numbers (or complex numbers) denoted with capital letters. $A_{m\times n} = a_{ij}$ is a matrix (named A) with m rows and n columns and *entries* a_{ij} where a_{ij} is the entry on the ith row in the jth column. A_n is a *square* matrix; that is, it has n rows and n columns.

I_n (or just I) is the $n \times n$ *identity matrix* with entries δ_{ij}, where $\delta_{ij} = 1$ if $i = j$ and is 0 otherwise. (The symbol δ_{ij} is the *Kronecker delta* symbol.) In other words, I has 1s along the main diagonal and 0s everywhere else.

A $1 \times n$ matrix is called a *row vector*; an $n \times 1$ matrix is a *column vector*.

Rules of Matrix Arithmetic (for matrices A, B, C and scalars a, b, c):

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $c(A + B) = cA + cB$
4. $(a + b)C = aC + bC$
5. $a(bC) = (ab)C$
6. $1A = A$
7. $0A = O$
8. $A + O = A$
9. $A + C = B$ iff $C = B - A$

Example 4.5. *Prove Rule 3 above.*

HOMEWORK FOR DAY 4. Page 17, #3, #13 parts c and d; Page 20, #2 (change directions to: identify which are not meaningful); Page 21, #3 part a, #4 part a, #5 part c

HOMEWORK ANSWERS. #3 see text

#13 c) $x = \frac{6}{-4} = -\frac{3}{2}, y = \frac{-8}{-4} = 2, z = \frac{-10}{-4} = \frac{5}{2}$.

#13 d) $x = t, y = -t, z = -2t$, for $t \in \mathbb{R}$.

#2 c and k are not meaningful

#3 a) $X = D - C = \begin{bmatrix} -1 & -4 \\ -2 & -1 \end{bmatrix}$

#4 a) $X = \frac{1}{2}(N+P) = \begin{bmatrix} \frac{3}{2} & 3 \\ -\frac{1}{2} & 1 \\ 5 & 2 \end{bmatrix}$ and $Y = N - X = \begin{bmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & 2 \\ 2 & -1 \end{bmatrix}$

#5 c)

$$\begin{aligned}(a+b)C &= \begin{bmatrix} (a+b)c_{11} & (a+b)c_{12} \\ (a+b)c_{21} & (a+b)c_{22} \end{bmatrix} \\ &= \begin{bmatrix} ac_{11} + bc_{11} & ac_{12} + bc_{12} \\ ac_{21} + bc_{21} & ac_{22} + bc_{22} \end{bmatrix} \\ &= \begin{bmatrix} ac_{11} & ac_{12} \\ ac_{21} & ac_{22} \end{bmatrix} \begin{bmatrix} bc_{11} & bc_{12} \\ bc_{21} & bc_{22} \end{bmatrix} = aC + bC\end{aligned}$$

5 Section 1.8, Matrix Multiplication

Objective. *Students will understand the defintion and use of matrix multiplication.*

Matrix multiplication comes from the *Cayley product* of linear transformations.

For example, a point $(1, -1)$ in the plane undergoes two transformations:

$$\begin{aligned} x' &= 2x - y \qquad & x'' &= x' - y' \\ y' &= x + 3y \qquad & y'' &= x' + 2y' \end{aligned}$$

Hence, $(1, -1) \Rightarrow (3, -2) \Rightarrow (5, -1)$. The net effect is obtained by the single transformation

$$\begin{aligned} x'' &= x - 4y \\ y'' &= 4x + 5y. \end{aligned}$$

Therefore we *define* multiplication so that

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 4 & 5 \end{bmatrix}$$

If $A_{m\times p}$ and $B_{p\times n}$ are multiplied, then the product is $(AB)_{m\times n}$. Hence, if A_n and a column vector $\mathbf{x} = X_{n\times 1}$ are multiplied, the result is another $n \times 1$ matrix (column vector).

Powers of matrices are defined in the usual way, $A^2 = AA$, $A^3 = AA^2$, etc.

Rules of Matrix Multiplication (for matrices A, B, C and scalars c, k, l):

10. $A(BC) = (AB)C$
11. $AI = A$
12. $IA = A$
13. $A(B + C) = AB + AC$
14. $c(AB) = A(cB)$
15. $AO = O$
16. $OA = O$
17. $A^0 = I$
18. $A^k A^l = A^{k+l}$
19. $(A^k)^l = A^{kl}$
20. $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$ iff $A = B$

HOMEWORK FOR DAY 5. Page 25, #1 (change directions to: identify which are not meaningful); Page 26, #4 part a, #6 (#8 extra credit)

HOMEWORK ANSWERS. #1 a, c, h, s are not meaningful

#4 a) $IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$

#6 Only when $AB = BA$.

#8 Assume all entries of B are distinct and that $A \neq O$. Then $AB = BA$ implies corresponding entries are equal:

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}.$$

Hence, the upper right entries are equal, as are the lower left entries:

$$a_{11}b_{12} + a_{12}b_{22} = b_{11}a_{12} + b_{12}a_{22}$$
$$a_{21}b_{11} + a_{22}b_{21} = b_{21}a_{11} + b_{22}a_{21}$$

Adding equations, setting equal to zero, and factoring, we have

$$(a_{11} - a_{22})(b_{12} - b_{21}) + (a_{12} - a_{21})(b_{22} - b_{11}) = 0.$$

But since all entries of B are distinct, it must be that $a_{11} = a_{22}$ and $a_{12} = a_{21}$. Note that the upper left entries are equal. This gives $a_{12}b_{21} = b_{12}a_{21}$; but since $a_{12} = a_{21}$, the only way this could be true is if $a_{12} = a_{21} = 0$. Hence, $a_{11} = a_{22} = c$ for some constant c. Therefore, $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI$.

6 Section 1.9, Matrix Inverses

Objective. *Students will use properties of matrix inverses to prove theorems involving matrix inverses.*

If $AB = I$, then $\det A \det B = 1$, or $\det A = \frac{1}{\det B}$.
$\det A \neq 0 \iff A^{-1}$ exists $\iff$ the system is solvable $\iff$ A is *nonsingular*.

Theorem 6.1. *If $AB = I$, then $BA = I$.*

Proof. $BA = BAI = BABB^{-1} = B(AB)B^{-1} = BIB^{-1} = BB^{-1} = I$. ☐

Theorem 6.2. *The inverse of a matrix is unique.*

Proof. Assume $AB = I$ and $AC = I$. Then by the theorem above, $BA = I$ and $CA = I$. Thus, $C = CI = CAB = IB = B$. ☐

Rules of Matrix Inverses (for matrices A, B and scalars c, p, q, r):

21. $(AB)^{-1} = B^{-1}A^{-1}$
22. $(cA)^{-1} = \frac{1}{c}A^{-1}$, for $c \neq 0$
23. $(A^{-1})^{-1} = A$
24. $(A^{-1})^p = A^{-p} = (A^p)^{-1}$
25. $(A^{p+q})^{-1} = A^{-p-q}$
26. $A^p A^q A^r = I$ iff $p + q + r = 0$

Example 6.1. *Simplify:* $[(AB)^{-1}A^{-2}]^{-1}$.

Systems of equations can be represented by $A\mathbf{x} = \mathbf{b}$. The solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 6.2. *Solve the systems from examples 5.1 through 5.4.*

Example 6.3. *Solve for X, and state which matrices are assumed to be nonsingular:* $X + Y = A, \quad X + BY = C$

HOMEWORK FOR DAY 6. Pages 30-31, #2 parts c, f, h, and j; #3 parts b and c; #4 part a; #8 parts c and d

HOMEWORK ANSWERS. #2 c) $\begin{bmatrix} \frac{107}{17} & -\frac{279}{34} \\ -\frac{32}{17} & \frac{89}{34} \end{bmatrix}$ f) $\begin{bmatrix} -\frac{43}{17} \\ \frac{27}{17} \end{bmatrix}$

#2 h) $\begin{bmatrix} -18 & 9 & -13 \\ 13 & -4 & 8 \end{bmatrix}$ j) $\begin{bmatrix} \frac{5}{2} & 1 & -\frac{3}{2} \end{bmatrix}$.

#3 b) $C^{-1}B^{-1}A^{-1}ABC = I$

#3 c) $\left[B^{-1}(A^{-1})^{-2}A^{-2}B^{-1}\right]^{-2} = (B^{-1}B^{-1})^{-2} = (B^{-2})^{-2} = B^4$

#4 a) $A^{-1}B^{-1} = (BA)^{-1} = (AB)^{-1} = B^{-1}A^{-1}$

#8 c) Subtract the two equations to get $(A - C)Y = B - D$. Therefore $Y = (A - C)^{-1}(B - D)$ and $X = B - AY$, assuming $A - C$ is nonsingular.

#8 d) Multiply the first equation by B^{-1} and the second by E^{-1}, then subtract to get

$$(B^{-1}A - E^{-1}D)X = B^{-1}C - E^{-1}F.$$

Thus, $X = (B^{-1}A - E^{-1}D)^{-1}(B^{-1}C - E^{-1}F)$ and $Y = B^{-1}(C - AX)$, assuming B, E, and $B^{-1}A - E^{-1}D$ are each nonsingular.

7 Section 1.10, Gaussian Elimination

Objective. *Students will use Gaussian elimination to solve systems of equations.*

To solve the system $A\mathbf{x} = \mathbf{b}$, form the *augmented matrix* $B = [A \quad \mathbf{b}]$. The following row operations are allowed to transform the matrix A to an *upper-triangular matrix*–a matrix with all zeros under the main diagonal.

i) adding multiples of one row to another row

ii) interchanging two rows

iii) multiplying a row by a nonzero scalar

As A is transformed, so is $\mathbf{b}$. By back-substitution, one obtains the solution.

If A is singular and $\mathbf{b} \neq \mathbf{0}$, then there is no solution, and one has r rows of nonzero entries in the upper-triangular matrix, with $n-r$ rows of zeros. The number r is the *rank* of the matrix. (Clearly, if $r = n$ then A is nonsingular.)

If A is singular and $\mathbf{b} = \mathbf{0}$, then there are infinite solutions.

Example 7.1. *Solve the systems in examples 5.1 through 5.4.*

Example 7.2. *Solve the system:* $\begin{cases} x - 3y + z & = 4 \\ -2x - 19y + 3z & = -3 \end{cases}$

HOMEWORK FOR DAY 7. Page 34, #3 parts a and b, #5 parts a and b, #6 part a

HOMEWORK ANSWERS. #3 a) The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the solutions are $x = 2,\quad y = t-1,\quad z = t$ for some parameter t.

#3 b) The augmented matrix is $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 5 \\ 0 & 2 & 0 & -1 & 1 \\ 0 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$, the solutions are $x = 4,\quad y = z = w = 1$.

#5 a) The augmented matrix is $\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & -5 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, the solutions are $x = \frac{1}{5}(7-t),\quad y = \frac{1}{5}(3t-1),\quad z = t$ for some parameter t.

#5 b) The augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{array}\right]$, and so there are no solutions.

#6 a) The augmented matrix is $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Multiplying the second row by $-\frac{a_1}{a_2}$ and adding the first and second results in $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & k_1 \\ 0 & b' & c' & k' \\ 0 & 0 & 0 & 0 \end{array}\right]$, where

$$b' = b_1 - \frac{a_1}{a_2}b_2,\ c' = c_1 - \frac{a_1}{a_2}c_2, \quad \text{and} \quad k' = k_1 - \frac{a_1}{a_2}k_2.$$

Letting $z = t$ gives

$$y = \frac{k' - c't}{b'} = \frac{a_2k_1 - a_1k_2 - (a_2c_1 - a_1c_2)t}{a_2b_1 - a_1b_2}$$

and

$$x = \frac{k_1 - b_1k'/b' + (b_1c'/b' - c_1)t}{a_1} = \frac{b_1k_2 - b_2k_1 + (b_2c_1 - b_1c_2)t}{a_2b_1 - a_1b_2}.$$

Hence, the solutions match the parametric form of a line in space given as Equation 1.28 on page 9.

8 Section 1.11A, Eigenvalues and Eigenvectors

Objective. *Students will find the eigenvalues of matrices by solving the characteristic polynomials.*

Notice: $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -2 \\ -5 & 3 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. So A and 3 have the same effect; i.e., there is a scalar λ and nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. The scalar λ is called an *eignevalue* and the vector $\mathbf{v}$ is called an *eigenvector*.

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \Rightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$

Finding $\det(A - \lambda I)$ results in a polynomial in λ, called the *characteristic polynomial*. The roots of $\det(A - \lambda I) = 0$ are the eigenvalues. The set of eigenvalues is the *spectrum*.

Example 8.1. *Find the eigenvalues and eigenvectors of* $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Answer: $\lambda_{1,2} = -1, 4$ and $\mathbf{v}_{1,2} = k_1\langle -1, 1\rangle, k_2\langle 2, 3\rangle$.

Example 8.2. *Find the eigenvalues and eigenvectors of* $\begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

Characteristic polynomial is $p(\lambda) = \lambda^3 + 8\lambda^2 + 19\lambda + 12$; eigenvalues are $\lambda_{1,2,3} = -1, -3, -4$. Eigenvectors are $\mathbf{v}_{1,2,3} = k_1\langle 1, 2, 1\rangle$, $k_2\langle 1, 0, -1\rangle$, $k_3\langle 1, -1, 1\rangle$.

Examining the char poly gives us another way to calculate eigenvalues that is quickly done for small matrices. Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and its char poly

$$p_A(\lambda) = \lambda^3 - b\lambda^2 + c\lambda - d = -\det(A - \lambda I) = 0$$

Note that b is the sum of the diagonal entries (called the *trace of* A), d is $\det A$, and c is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

These smaller determinants are the *princpal minors of* A.

Example 8.3. *Find the eigenvalues and eigenvectors of* $A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}$

We see that the trace of A is -8; $\det A = -12$, and the sum of the principal minors is $5 + 9 + 5 = 19$. Hence the char poly is $p(\lambda) = \lambda^3 + 8\lambda^2 + 19\lambda + 12$ as before.

Clearly it is possible to have imaginary eigenvalues; what ensures real eigenvalues? We introduce the *transpose* of a matrix: The transpose A^T of a matrix A is obtained by interchanging rows and columns. A matrix A is *symmetric* if $A = A^T$. All eigenvalues of a symmetric matrix are real. Moreover, we the following:

Theorem 8.1. *Every $n \times n$ symmetric matrix has n mutually orthogonal eigenvectors.*

Example 8.4. *Verify that the eigenvectors of Example 8.2 are mutually orthogonal.*

Example 8.5. *Find the eigenvalues and eigenvectors of* $B = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

The char poly is $p_B(\lambda) = (\lambda + 2)^2(\lambda - 4)$, the eigenvalues are $\lambda_{1,2,3} = -2, -2, 4$. The eigenvector for $\lambda = -2$ is a linear combination: $\mathbf{v}_{1,2} = k_1\langle -1, 1, 0\rangle + k_2\langle -1, 0, 1\rangle$. The remaining eigenvector is $\mathbf{v}_3 = k_3\langle 1, 1, 1\rangle$. But the three vectors are not mutually orthogonal since there is a repeated eigenvalue; to find the third orthogonal vector, we compute the cross product of either v_1 and v_3 or of v_2 and v_3.

HOMEWORK FOR DAY 8. Page 38, #1, #2 part a, #4 parts c and d

HOMEWORK ANSWERS. #1 b) Characteristic polynomial is $\lambda^2 - 7\lambda$, eigenvalues are 0 and 7, eigenvectors are $k\langle -3, 1\rangle$ and $k\langle 1, 2\rangle$.

#1 c) Characteristic polynomial is $-\lambda^3 + 6\lambda^2 - 11\lambda + 6$, eigenvalues are 1, 2, and 3, eigenvectors are $k\langle 0, 2, 1\rangle$, $k\langle 1, 2, 0\rangle$, and $k\langle 1, 1, -1\rangle$.

#4 c) Characteristic polynomial is $(\lambda + 1)^2$, eigenvalue is -1, eigenvector is $k\langle 1, 1\rangle$.

9 Section 1.11B, Similar Matrices

Objective. *Students will determine whether two matrices are similar.*

Further properties of eigenvalues:

- A and A^T have the same eigenvalues.
- If $p(x)$ is any polynomial and λ is an eigenvalue of A, then $p(\lambda)$ is an eigenvalue of $p(A)$.
- A matrix is nonsingular iff its eigenvalues are all nonzero.
- If A is nonsingular, then the eigenvalues of A^{-1} are reciprocals of the eigenvalues for A.

For matrices A, B, if there exsits a nonsingular C such that $B = C^{-1}AC$, then B is similar to A; since $A = (C^{-1})^{-1}BC^{-1}$, then A is similar to B as well.

Theorem 9.1. *If A and B are similar, they have the same characteristic polynomial.*

Proof.

$$\begin{aligned}\det(B-\lambda I) &= \det(C^{-1}AC-\lambda I) = \det(C^{-1}AC-\lambda C^{-1}IC)\\ &= \det C^{-1}(A-\lambda I)C = \det C^{-1}\det(A-\lambda I)\det C\\ &= \det(A-\lambda I)\end{aligned}$$

since $\det C^{-1} = \frac{1}{\det C}$. □

Thus, if A and B are similar, they have the same eigenvalues; also, A is similar to the diagonal matrix of its eigenvalues, denoted $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Spectrum of A may include k zeros, in which case the rank is equal to $n-k$; i.e., A has k rows of zeros in its row echelon form.

HOMEWORK FOR DAY 9. Page 38, #6 part a, #7; Page 39, #8

HOMEWORK ANSWERS. #6 a) A similar to $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ implies that there exists a nonsingular C such that $A = C^{-1}\operatorname{diag}(\lambda_1, \ldots, \lambda_n)C$. Thus

$$\begin{aligned}
\det A &= \det[C^{-1}\operatorname{diag}(\lambda_1, \ldots, \lambda_n)C] \\
&= \det C^{-1} \det[\operatorname{diag}(\lambda_1, \ldots, \lambda_n)] \det C \\
&= \det[\operatorname{diag}(\lambda_1, \ldots, \lambda_n)] \\
&= \lambda_1 \cdots \lambda_n
\end{aligned}$$

#8 a) Since $A = IAI = I^{-1}AI$, A is similar to itself.

#8 b) Since A and B are similar, there is nonsingular P such that $A = P^{-1}BP$. Since B and C are similar, there is nonsingular Q such that $B = Q^{-1}CQ$. Hence,

$$A = P^{-1}BP = P^{-1}Q^{-1}CQP = (QP)^{-1}C(QP).$$

Since there is nonsingular QP such that $A = (QP)^{-1}C(QP)$, then A and C are similar.

10 Section 1.13, Orthogonal Matrices

Objective. *Students will define and prove theorems involving properties of the transpose of a matrix, symmetric matrices, and orthogonal matrices.*

Rules of the Transpose (for matrices A, B and scalar c):

24) $(A+B)^T = A^T + B^T$

25) $(cA)^T = cA^T$

26) $(A^T)^T = A$

27) $(AB)^T = B^T A^T$

28) If A is nonsingular, $(A^{-1})^T = (A^T)^{-1}$

29) $\det A = \det A^T$

A matrix A is *orthogonal* if $AA^T = I$, or $A^T = A^{-1}$. So orthogonal implies nonsingular. For example: $A = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ -\frac{12}{13} & \frac{5}{13} \end{bmatrix}$ is orthogonal.

Theorem 10.1. *Let A be an $n \times n$ orthogonal matrix. Then each row vector of A is a unit vector.*

Proof. Since $AA^T = I$, we have, by matrix multiplication, that

$$a_{i1}^2 + \cdots + a_{in}^2 = 1.$$

Thus, $\langle a_{i1}, \ldots, a_{in} \rangle$ is a unit vector. □

Theorem 10.2. *Let A be an $n \times n$ orthogonal matrix. Then different rows are orthogonal.*

Proof. Since $AA^T = I$, we have, by matrix multiplication, that

$$a_{i1}a_{j1} + \cdots + a_{in}a_{jn} = 0.$$

Thus, $\langle a_{i1}, \ldots, a_{in} \rangle \cdot \langle a_{j1}, \ldots, a_{jn} \rangle = 0$ and different row vectors are orthogonal. □

Orthogonal matrices whose determinants are 1 are also known as *rotational* matrices. For example, the matrix

$$\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$

is the standard rotational matrix: Let $\omega = \arctan\frac{12}{5}$ and we get the orthogonal matrix above. To rotate points 90° we use the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

HOMEWORK FOR DAY 10. Page 44, #1, #2, #5 part b, #8

HOMEWORK ANSWERS. #2 a) Set $3a - 1 = 2a$ to get $a = 1$.

#2 b) $a = b - a,\ b = 4 + a$ implies $a = 4, b = 8$.

#5 b)

$$\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\omega + \sin^2\omega & 0 \\ 0 & \sin^2\omega + \cos^2\omega \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

#8 a) $AA^T = I \Rightarrow \det(AA^T) = \det I \Rightarrow \det A \det A^T = 1 \Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1$, since $\det A = \det A^T$.

#8 b) $I = AA^T = AIA^T = ABB^TA^T = AB(AB)^T$.

#8 c) $I = AA^T \Rightarrow A^{-1} = A^T \Rightarrow A^{-1}A = A^TA = I \Rightarrow A^T(A^T)^T = I$. Next, $I = AA^T \Rightarrow I = (AA^T)^{-1} = (A^T)^{-1}A^{-1} = (A^{-1})^TA^{-1}$.

11 Sections 1.12 and 1.13, Quadratic Forms

Objective. *Students will determine coefficient matrices for quadratic forms and use the eigenvalues to determine the new coordinates of rotation.*

A quadratic form is any sum of quadratic terms; we consider the 2-dimensional quadratic form $-d = ax^2 + 2bxy + cy^2$. The presence of the xy term indicates that the conic section this represents has been rotated so that the x and y axis are no longer the principal axes. We use a coefficient matrix A and the variable vector $\mathbf{v} = \langle x, y\rangle$ to write the equation as

$$-d = \mathbf{v}^T A\mathbf{v} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We then use a rotational matrix Q to denote the change in axes: $\mathbf{v} = Q\mathbf{v}'$. Then

$$-d = \mathbf{v}^T A\mathbf{v} = (Q\mathbf{v}')A(Q\mathbf{v}') = \mathbf{v}'^T(Q^TAQ)\mathbf{v}' = \begin{bmatrix} x' & y' \end{bmatrix} (Q^TAQ) \begin{bmatrix} x' \\ y' \end{bmatrix}$$

This is great, but how do we know what rotation to use? The signal for this is that the $2b'x'y'$ must disappear; so we define a diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ so that $Q^TAQ = D$. So starting from A, we need to solve the equation $Q^TAQ = D$ for Q and D; but this is easy: the column vectors of Q are the eigenvectors of A and the diagonal elements of D are the eigenvalues of A! Moreover, the directions of the new axes are the eigenvectors!

Example 11.1. *Determine the standard form for $xy = 1$.*

The quadratic form is $1 = 0x^2 + xy + 0y^2$, so $A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$. The eigenvalues are $\lambda = \pm\frac{1}{2}$, the eigenvectors are $\mathbf{v} = \langle 1, \pm 1\rangle$. Hence,

$$\begin{aligned} \begin{bmatrix} x' & y' \end{bmatrix} D \begin{bmatrix} x' \\ y' \end{bmatrix} &= \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \\ &= \frac{1}{2}x'^2 - \frac{1}{2}y'^2 = 1 \end{aligned}$$

or, $x'^2 - y'^2 = 2$ with x'-axis $\langle 1, 1\rangle$ and y'-axis $\langle 1, -1\rangle$. Finally we have the actual rotation by computing Q. The column vectors must be unit vectors since $\det Q = 1$. Hence, $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. These are sine and cosine values, so taking $\arccos \frac{1}{\sqrt{2}}$ we have a rotation from the standard axis of $45°$.

Example 11.2. *Determine the standard form for* $x^2 + 4xy - 2y^2 = 6$.

We have $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$; eigenvalues -3 and 2; and eigenvectors $\langle 1, -2 \rangle$ and $\langle 2, 1 \rangle$. Hence, the standard form is $2x'^2 - 3y'^2 = 6$ with an angle of rotation of $\arccos \frac{2}{\sqrt{5}} \approx 26.5°$.

Quadratic forms with three terms are done similarly; the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

corresponds to the form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = -c$$

HOMEWORK FOR DAY 11. Page 44, #3

12 Section 1.14, Vectors in *n*-Dimensional Space

Objective. *Students will determine the linear independence of n vectors and extend known results to n dimensions. Students will prove the Cauchy-Schwartz and Triangle Inequalities.*

Discuss "Euclidean space"; note properties on page 47. Read "Remarks" on page 47 concerning linear independence: if matrix A consists of column vectors and $\det A \neq 0$, then the vectors are linearly independent. Numerous properties of linear independence; page 48.

Discuss basis for a vector space and an orthogonal basis in particular. An orthogonal basis of unit vectors is an orthonormal basis.

Theorem 12.1 (Cauchy-Schwartz Inequality). *For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$*,*

$$||\mathbf{u}\cdot\mathbf{v}|| \leq ||\mathbf{u}||||\mathbf{v}||.$$

Proof. Without loss of generality, assume $\mathbf{v} \neq 0$ and $\mathbf{u} \neq k\mathbf{v}$ for some real k. Then $\mathbf{u} + t\mathbf{v} \neq \mathbf{0}$ for all real t. Thus, $||\mathbf{u}+t\mathbf{v}||^2 > 0$, so that

$$(\mathbf{u}+t\mathbf{v})(\mathbf{u}+t\mathbf{v}) > 0 \Rightarrow ||\mathbf{u}||^2 + 2t(\mathbf{u}\cdot\mathbf{v}) + t^2||\mathbf{v}||^2 > 0.$$

Hence, $4(\mathbf{u}\cdot\mathbf{v})^2 - 4||\mathbf{u}||^2||\mathbf{v}||^2 < 0$ since there can be no real zeros. Therefore, $||\mathbf{u}\cdot\mathbf{v}|| < ||\mathbf{u}||||\mathbf{v}||$.

If $\mathbf{u} = k\mathbf{v}$, then

$$||\mathbf{u}\cdot\mathbf{v}|| = ||k\mathbf{v}\cdot\mathbf{v}|| = ||k||||\mathbf{v}||^2 = ||k\mathbf{v}||||\mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}||.$$

□

Theorem 12.2 (Triangle Inequality). *For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$*,*

$$||\mathbf{u}+\mathbf{v}||^2 \leq (||\mathbf{u}|| + ||\mathbf{v}||)^2.$$

Proof.

$$\begin{aligned} ||\mathbf{u}+\mathbf{v}||^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 + 2\mathbf{u}\cdot\mathbf{v} \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 + 2\mathbf{u}\cdot\mathbf{v} - 2||\mathbf{u}||||\mathbf{v}|| \\ &\leq (||\mathbf{u}|| + ||\mathbf{v}||)^2. \end{aligned}$$

□

HOMEWORK FOR DAY 12. Page 53, #1, #5 part a

13 Section 1.16, *n*-Dimensional Linear Mappings

Objective. *Students will understand the concept of a linear mapping and its properties.*

A *function* is the assignment to each object in a first set (called the *domain*) an object in the second set (called the *codomain*). The *range* is the set of objects in the codomain that are actually used. A function is also called a *mapping*. The function "maps" the domain *into* the codomain; when the range is the same as the codomain, the function maps *onto* the range.

A mapping (or, function) T is a *linear mapping* if, there exists a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x}$ in the domain. Notation: "$T : V^n \to V^m$" is read *"T maps a vector from an n-dimensional vector space to a vector in an m-dimensional vector space."*

For a mapping $T : V^n \to V^m$, we are concerned with

- the range of T,
- whether T maps V^n onto V^m,
- whether T is one-to-one, and
- the *kernel* of T—the set of all $\mathbf{x}$ for which $T(\mathbf{x}) = \mathbf{0}$

Theorem 13.1. *Let $T : V^n \to V^m$ be a linear mapping. Then T is one-to-one if and only if the kernel of T consists of the zero vector alone.*

If $T(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x}$, then T is the *zero mapping*. This is neither one-to-one nor onto.

If $T(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x}$, then T is the *identity mapping*. This is one-to-one and onto.

Examples 4 and 7, page 59.

HOMEWORK FOR DAY 13. Page 60, #1, #3, #8

HOMEWORK ANSWERS. #1 a) $T(1,0) = (2,3)$; $T(0,1) = (1,5)$; $T(2,-1) = (3,7)$; $T(-1,1) = (-1,2)$. b) $A\mathbf{x} = \mathbf{0}$ implies $2x_1 + x_2 = 0$ and $3x_1 + 5x_2 = 0$, whose only solution is $x_1 = x_2 = 0$; hence, only $\mathbf{0}$ is in the kernel, so T is one-to-one. c) Range is V^2 and T is onto.

#3 a) $n = 3$, $m = 2$. b) $t\langle 0, 2, -1\rangle$ for real t; not one-to-one c) V^2; onto

#8 a) $n = m = 3$ b) $\mathbf{0}$; one-to-one c) V^3; onto

14 Section 2.2, Domains and Regions

Objective. *Students will understand set-theoretic terminology as applied to functions.*

A *set of points* is any collection of points in the xy-plane whether finite or infinite.

A *neighborhood* of a point (x_0, y_0) is a set of points inside a circle of radius δ with center (x_0, y_0). Each point (x, y) in the neighborhood satisfies $(x - x_0)^2 + (y - y_0)^2 < \delta^2$.

A set of points is *open* if every point of the set has a neighborhood lying within the set. These are defined by strict inequalities.

A set of points is *closed* if the set includes the interior points and the points lying on the boundary; i.e., set S is closed if the points in the plane that are not in S form an open set.

A set is *bounded* if there exists a circle of large enough radius to enclose the entire set.

An open set is a *connected open set* is any two points A and B can be connected by "broken" line segments. Also known as a *domain*.

A *boundary point* of a set is a point whose every neighborhood consists of points inside and outside the set.

An *interior point* of a set has a neighborhood that is entirely contained within the set. —[[THOMAS 11.9]]—

A *region* is a domain plus some, none, or all of its boundary points. A domain and its boundary is called a closed region. A domain is also called an open region.

Example 14.1. *We clarify the above definitions with a few examples.*

- *the set* $xy < 1$ *is a domain;*
- *the set* $xy < 1$ *has boundary* $xy = 1$*;*
- *the set* $xy \leq 1$ *is a closed region;*
- *the set* $xy \leq 1$ *is not bounded;*
- *the point* $(2, \frac{1}{2})$ *is a boundary point of* $xy < 1$ *and* $xy \leq 1$*;*
- *the point* $(1, \frac{1}{2})$ *is an interior point of* $xy < 1$ *and* $xy \leq 1$*.*

15 Section 2.3, Functions and Level Curves

Objective. *Students will sketch the level curves of a two-variable function. Students will recognize equations of various quadric surfaces. Students will determine the domain of a two-variable function.*

—[[THOMAS 10.3]]— Function $z = f(x,y)$ is defined by $z = \sqrt{1-x^2-y^2}$. The domain is the closed set $x^2+y^2 \leq 1$. Note $f(0,0)=1$ and $f(\frac{1}{2},\frac{1}{2}) = \sqrt{\frac{1}{2}}$.

Graph $z = f(x,y)$ by sketching *level curves*: plot $f(x,y) = c$ for choices of c. —[[THOMAS 11.4, 11.5, LARSON 37]]—

Another graphing method is is to keep one variable fixed and find "traces" in each coordinate plane. —[[STEWART 32, LARSON 97]]—

Example 15.1. *Find the family of level curves for* $z = \sqrt{1-x^2-y^2}$.

Let $c = 0$ and graph the resulting curve $y = \sqrt{1-x^2}$. Choose $c = \sqrt{\frac{1}{2}}$ and graph $y = \sqrt{\frac{1}{2}-x^2}$. The level curves are all circles with radius less than 1 and centered at the origin.

Level curves are like topographic maps or isotherms. —[[STEWART 36]]—

Three-dimensional graphs. —[[STEWART 35, 38]]—

Example 15.2. *Match the 3-D graphs with their level curves; match the 3-D graphs with their equations.*

—[[STEWART 40, 30]]—

Quadric Surfaces —[[LARSON 87, 88]]—

Ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$; a sphere if $a=b=c$. Simpler: $x^2+y^2+z^2=k$

Hyperboloid of One Sheet $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$; axis corresponds to negative variable. Simpler: $x^2+y^2=z^2+k$

Hyperboloid of Two Sheets $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$; axis corresponds to positive variable. Simpler: $x^2-k=y^2+z^2$

Elliptic Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$; axis corresponds to negative variable. Simpler: $x^2 + y^2 = z^2$

Elliptic Paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$; axis corresponds to linear variable. Simpler: $z = x^2 + y^2$

Hyperbolic Paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$; axis corresponds to linear variable. Simpler: $z = x^2 - y^2$

Example 15.3. *Find the domains of the following functions.*

a) $f(x, y) = \dfrac{\sqrt{x^2 + y^2}}{x^2 + 3x - 8}$

b) $f(x, y) = -2\cos(2x) + y$

c) $f(x, y) = \sin\sqrt{1 - (x^2 + y^2)}$

d) $f(x, y) = \exp\left(\dfrac{x + y}{xy}\right)$

HOMEWORK FOR DAY 15. Page 82, #2

HOMEWORK ANSWERS. #2

a) Sketch $y = \frac{1}{3}(c - 3 + x)$; curves are lines with slope $\frac{1}{3}$.

b) Sketch $y = \pm\sqrt{c - x^2 - 1}$; curves are circles with radius $\delta \geq 1$.

c) Sketch $y = \arcsin c - x$; curves are lines with slope -1.

d) Sketch $y = \frac{\ln z}{x}$; curves are hyperbolas with asymptotes at the x, y-axes.

16 Section 2.4, Continuity, Limits, and Mappings

Objective. *Students will define the limits of a two-variable function and determine whether a two-variable function is continuous. Students will use properties of limits to evaluate limits. Students will determine a metric of a Euclidean space as it applies to linear mappings.*

Definition of a Limit:

Let $f(x, y)$ have domain D. If $(x_1, y_1) \in D$ such that

$$0 < (x - x_1)^2 + (y - y_1)^2 < \delta^2$$

then there is ε such that

$$|f(x, y) - L| < \varepsilon.$$

The number L is the limit, denoted $\lim\limits_{\substack{x \to x_1 \\ y \to y_1}} f(x, y) = L$.

If the limit at a point equals the function value at that point for every point in the domain, then the function is continuous.

Example 16.1. *Is* $z = \dfrac{x - y}{x + y}$ *continuous at* $(0, 0)$*?*

Along the line $y = 0$, $z = 1$, so the limit is 1.

Along the line $x = 0$, $z = -1$, so the limit is -1.

Along the line $y = x$, $z = 0$, so the limit is 0.

Since the limit is different for each case, the function is discontinuous at the origin.

Theorem 16.1. *Let* $f(x, y)$ *and* $g(x, y)$ *be defined in the same domain* D *and let*

$$\lim_{\substack{x \to x_1 \\ y \to y_1}} f(x, y) = u, \lim_{\substack{x \to x_1 \\ y \to y_1}} g(x, y) = v.$$

Then

$$\lim_{\substack{x\to x_1\\y\to y_1}}[f(x,y)+g(x,y)]=u+v$$
$$\lim_{\substack{x\to x_1\\y\to y_1}}[f(x,y)\cdot g(x,y)]=u\cdot v$$
$$\lim_{\substack{x\to x_1\\y\to y_1}}\frac{f(x,y)}{g(x,y)}=\frac{u}{v}\quad(v\neq 0)$$
$$\lim_{\substack{x\to x_1\\y\to y_1}}F\left(f(x,y),g(x,y)\right)=F(u,v)$$

Also, if f and g are continuous, so are the functions $f+g$, $f\cdot g$, f/g, and $F(f,g)$.

Example 16.2. *Find $\lim_{\substack{x\to 0\\y\to 0}} f(x,y)$ for each function.*

a) $f(x,y)=\dfrac{xy}{x+y+1}$

b) $f(x,y)=\dfrac{x^2+y^2}{x}, f(0,y)=0$

c) $f(x,y)=\dfrac{x^2-y^2}{x-y}, f(x,x)=0$

d) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$

e) $f(x,y)=\dfrac{\sin(x^2+y^2)}{x^2+y^2}$, $f(0,0)=0$

f) $f(x,y)=e^{x^2+y^2}$

g) $f(x,y)=\dfrac{x+y-\sin(x+y)}{(x+y)^3}$

h) $f(x,y)=\dfrac{xy}{x^3-y^3}, f(x,x)=0$

Example 16.3. *Which of the functions in the previous example are continuous at $(0,0)$?*

Interpretations of mappings from V^n to V^m:

a set of functions $y_m=f_m(x_1,\ldots,x_n)$, vector functions $\mathbf{y}=\mathbf{f}(\mathbf{x})$, or points in space with a defined distance, or *metric* function; i.e., $A=(x_1,\ldots,x_n)\in E^n$ and $B=(y_1,\ldots,y_m)\in E^m$ with $d(A,B)$ defined as the distance between A and B. Then instead of vector notations, we can use a multivariable function F such that $F(A)=B$. This gives us a new definition of continuity:

F is continuous at $P_0\in E^n$ if for each $\varepsilon>0$ there is a $\delta>0$ such that, for $P\in E^n$, $d(F(P),F(P_0))<\varepsilon$ whenever $d(P,P_0)<\delta$.

The Euclidean metric has the following properties:

i) $d(A, B) \geq 0$

ii) $d(A, B) = d(B, A)$

iii) $d(A, C) \leq d(A, B) + d(B, C)$

Vector spaces have a metric: $||\mathbf{a} - \mathbf{b}||$.

Example 16.4. *Prove that* $f : V^n \to \mathbb{R}, f(\mathbf{x}) = ||\mathbf{x}||$ *is continuous.*

HOMEWORK FOR DAY 16. Page 82, #4, #5, #6

HOMEWORK ANSWERS. #4

a) 0

b) does not exist

c) 1

d) ∞

#5 a) Discontinuous at the origin since the limit is undefined along $y = x$, 1 along $y = 0$, and 0 along $x = 0$.

#5 b) Discontinuous at the origin since the limit is $-\infty$.

#6

a) Defined on the domain of all (x, y)

b) Defined on the domain of the exterior of a circle of radius 1

c) Defined on the closed region of the circle of radius 1

d) Defined on open region of all space except the xy-plane

17 Section 2.5, Partial Derivatives

Objective. *Students will compute partial derivatives using basic differentiation rules.*

Let y be fixed, say, at y_0. Then $f(x, y_0)$ depends only on x so we have

$$\frac{\partial f}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y_0) - f(x, y_0)}{\Delta x}$$

as the *partial derivative of f with respect to x.* One can then evaluate this derivative at the point (x_0, y_0).

For $z = f(x,y)$, we have the notations $\frac{\partial f}{\partial x}$, $\frac{\partial z}{\partial x}$, and f_x. —[[Larson 99]]—

Geometrically, $\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}$ is the slope of the tangent line to the surface f in the plane $y = y_0$. —[[Stewart 41]]—

The partial with respect to y is defined analogously. Sometimes we specify which variables are kept constant:

$$\left(\frac{\partial w}{\partial x}\right)_{yz}$$

means $f_x(x,y,z)$ for the function $w = f(x,y,z)$. —[[Thomas 11.13, 11.14, 11.15]]—

Any number of variables are easily defined:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{\Delta x_i \to 0} \frac{f(x_1,\ldots,x_i+\Delta x,\ldots,x_n) - f(x_1,\ldots,x_i,\ldots,x_n)}{\Delta x}$$

Example 17.1. *Find both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.*

a) $f(x,y) = 5x^2y$

b) $f(x,y) = y\cos x$

c) $f(x,y) = 3x^2 + xy^2$

d) $f(x,y) = \dfrac{x^2+y^2}{x}$

e) $z = \sqrt{x^2+y^2}$

f) $x^2 + y^2 - z^2 = 1$

g) $x^2 + y + z + \log z = 2$

h) $x \log \frac{y}{z} = 1$

Example 17.2. *Find both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the indicated point.*

i) $f(x,y) = \sin x \cos y$ *at* $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

ii) $f(x,y) = x\arctan(xy)$ *at* $(1,0)$.

iii) $f(x,y) = x^2y + y^2z + z^2x$ *at* $(1,2,3)$.

Example 17.3. *Suppose* u, v, x, y *are related by* $u = x^2 - y,\ v = x - 2y^2$. *Then* $(\frac{\partial u}{\partial y})_x = -1$ *and* $(\frac{\partial v}{\partial x})_y = 1$

Example 17.4. *Example 2, page 85*

HOMEWORK FOR DAY 17. Page 89, #1, #3

HOMEWORK ANSWERS. $\#1$

a) $\dfrac{\partial z}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2}$ and $\dfrac{\partial z}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

b) $\dfrac{\partial z}{\partial x} = y^2 \cos xy$ and $\dfrac{\partial z}{\partial y} = \sin xy + xy\cos xy$

c) $\dfrac{\partial z}{\partial x} = \dfrac{2xz - 3x^2 - 2xy}{3z^2 - x^2}$ and $\dfrac{\partial z}{\partial y} = \dfrac{x^2}{x^2 - 3z^2}$

d) $\dfrac{\partial z}{\partial x} = \dfrac{e^{x+2y}}{2\sqrt{e^{x+2y} - y^2}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{e^{x+2y} - y}{\sqrt{e^{x+2y} - y^2}}$

e) $\dfrac{\partial z}{\partial x} = 3x\sqrt{x^2+y^2}$ and $\dfrac{\partial z}{\partial y} = 3y\sqrt{x^2+y^2}$

f) $\dfrac{\partial z}{\partial x} = \dfrac{1}{\sqrt{1-(x+2y)^2}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{2}{\sqrt{1-(x+2y)^2}}$

g) $\dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^z - 1}$ and $\dfrac{\partial z}{\partial y} = \dfrac{2e^y}{e^z - 1}$

$\#3$ c) $\dfrac{\partial x}{\partial u} = 1$, $\dfrac{\partial y}{\partial v} = -\frac{1}{2}$

18 Section 2.6, The Total Differential

Objective. *Students will compute the total differential of a two-variable function. Students will understand and apply the Fundamental Lemma.*

Last time, partial derivatives were found by changing x and y separately. What is the effect of changing them together? We have —[[Larson 100]]—

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y).$$

Example 18.1. *If* $f(x, y) = x^2 + xy$, *then*

$$\begin{aligned}\Delta z &= (x + \Delta x)^2 + (x + \Delta x)(y + \Delta y) - (x^2 + xy)\\ &= x^2 + 2x\Delta x + (\Delta x)^2 + xy + x\Delta y + y\Delta x + \Delta x\Delta y - x^2 - xy\\ &= (2x + y)\Delta x + x\Delta y + (\Delta x)^2 + \Delta x\Delta y\end{aligned}$$

In the example, we have

$$\Delta z = a\Delta x + b\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \tag{18.1}$$

where a and b are functions independent of Δx and Δy, and ε_1 and ε_2 are functions such that

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_1 = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_2 = 0.$$

Thus, the function $a\Delta x + b\Delta y = dz$ is the total differential of z and approximates Δz for small values of Δx and Δy.

Theorem 18.1. *If* $z = f(x, y)$ *has a total differential at the point* (x, y), *then* f *is continuous at* (x, y) *and* $a = \dfrac{\partial z}{\partial x}$, $b = \dfrac{\partial z}{\partial y}$.

Proof. Let $\Delta y = 0$. Then in Eq. 18.1 we have

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x(a + \varepsilon_1)}{\Delta x} = \lim_{\Delta x\to 0} (a + \varepsilon_1) = a.$$

Similarly, $\frac{\partial z}{\partial y} = b$. Since $\Delta z \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, f is continuous at (x, y). □

Note: this theorem implies that the total differential, when existent, is *unique*. Also, the existence of the partials at the point is *not enough* to guarantee the existence of the total differential!

Theorem 18.2 (Fundamental Lemma). *If $z = f(x, y)$ has continuous first partial derivatives in domain D, then z has a differential*

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

at every point of D.

This is easily generalizable to three or more variables.

Example 18.2. *If $z = x^2 - y^2$, then $dz = 2x\ dx - 2y\ dy$.*

Example 18.3. *If $w = x \sin y - y \cos z + z \tan x$, then*

$$dw = (\sin y + z \sec^2 x)\ dx + (x \cos y - \cos z)\ dy + (y \sin z + \tan x)\ dz.$$

Example 18.4. *If $z = f(x, y) = 3x^2 - xy$, use Δz to approximate the change in z from $(1, 2)$ to $(1.01, 1.98)$.*

Example 18.5. *If $w = f(x, y, z) = xyz$, use Δw to approximate the change in w from $(9, 6, 4)$ to $(9.02, 5.97, 4.01)$.*

The previous example could represent the approximate change in the volume of a box given slight distortion of its side lengths.

HOMEWORK FOR DAY 18. Page 89, #4; Page 90, #5

HOMEWORK ANSWERS. #4

a) $dz = \dfrac{1}{y}\ dx - \dfrac{x}{y^2}\ dy = \dfrac{z}{x}\ dx - \dfrac{z}{y}\ dy$

b) $dz = \dfrac{x}{x^2+y^2}\ dx + \dfrac{y}{x^2+y^2}\ dy$

c) $dz = \dfrac{y+1}{(1-x-y)^2}\ dx + \dfrac{x+1}{(1-x-y)^2}\ dy$

d) $dz = (x-2y)^4 e^{xy}\left[(xy-2y^2+5)\ dx + (x^2-2xy-10)\ dy\right]$

e) $dz = \dfrac{-y/x^2}{1+y^2/x^2}\ dx + \dfrac{1/x}{1+y^2/x^2}\ dy = \dfrac{-y}{x^2+y^2}\ dx + \dfrac{x}{x^2+y^2}\ dy$

f) $du = (x^2+y^2+z^2)^{-3/2}(-x\ dx - y\ dy - z\ dz) = -u^3(x\ dx + y\ dy + z\ dz)$

#5 a) First, we have

$$\Delta z = x^2 + 2x\Delta x + (\Delta x)^2 + 2xy + 2x\Delta y + 2y\Delta x + 2\Delta x\Delta y - x^2 - 2xy$$

so that

$$\Delta z|_{(1,1)} = 4\Delta x + (\Delta x)^2 + 2\Delta y + 2\Delta x\Delta y$$

Next, we have $dz = (2x+2y)\ dx + 2\ dy$ so that $dz|_{(1,1)} = 4\ dx + 2\ dy$. At $\Delta x = \Delta y = dx = dy = 0.01$, i.e., at $x = y = 1.01$, we have

$$\Delta z = 4(0.1) + (0.1)^2 + 2(0.1) + 2(0.1)^2 = 0.0603$$

and $dz = 4(0.1) + 2(0.1) = 0.06$.

19 Section 2.7, The Jacobian

Objective. *Students will use the Jacobian to approximate a multivariable function at a point.*

The differential of n variables of a function $y = f(x_1, \cdots, x_n)$ is $dy = f_{x_1}dx_1 + \cdots + f_{x_2}dx_n$, whose coefficients are the partials. We could have system of m equations, each of n-variables on $D \subseteq E^n \to E^m$. To clarify this, we use two equations, each of three variables. The system of differentials is then

$$\begin{cases} dy_1 = \frac{\partial f_1}{\partial x_1}dx_1 + \frac{\partial f_1}{\partial x_2}dx_2 + \frac{\partial f_1}{\partial x_3}dx_3 \\ dy_2 = \frac{\partial f_2}{\partial x_1}dx_1 + \frac{\partial f_2}{\partial x_2}dx_2 + \frac{\partial f_2}{\partial x_3}dx_3 \end{cases}$$

or,

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

or,

$$\mathrm{col}\langle dy_1, dy_2 \rangle = \left(\frac{\partial f_i}{\partial x_j} \right) \mathrm{col}\langle dx_1, dx_2, dx_3 \rangle$$

where $i = 1, 2$ and $j = 1, 2, 3$. The matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ is called the *Jacobian* matrix where entries are partials evaulated at a point.

The system is a mapping from $D \subseteq E^3$ to E^2. The matrix equation is a linear mapping that approximates the mapping near a point; so we have $d\mathbf{y} = \mathbf{f_x} d\mathbf{x}$ where $\mathbf{f_x}$ is the Jacobian. —[[Stewart 44]]—

Example 19.1. *Find the Jacobians for a)* $y_1 = 5x_1 + 2x_2,\ y_2 = 2x_1 + 3x_2$ *and b)* $y_1 = 2x_1^2 + x_2^2,\ y_2 = 3x_1x_2$.

Example 19.2. *Find the Jacobian for* $y_1 = x_1^2 + x_2^2 - x_3^2,\ y_2 = x_1^2 - x_2^2 + x_3^2,\ y_3 = -x_1^2 + x_2^2 + x_3^2$ *at the point* $(2, 1, 1)$, *and use it to approximate* $\mathbf{y}$ *at* $(2.01, 1.03, 1.02)$. *(Example 1, page 92)*

If $m = n$, then the Jacobian is square, and we may find its determinant J. This measures the ratio of the n-dimensional volumes.

Example 19.3. *Find the Jacobian and its determinant for* $u = x^2 - xy$ *and* $v = xy + y^2$ *at the point* $(1, 1)$, *and use it to approximate* (u, v) *at* $(1.01, 1.02)$.

For $1 \leq x, y \leq 2$, the square in the xy-plane is mapped to a curved parallelogram in the uv-plane with vertices $(0,2)$, $(2,3)$, $(0,8)$, $(-1,6)$. The Jacobian is

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} (2x-y)\ dx & -x\ dy \\ y\ dx & (x+2y)\ dy \end{bmatrix}$$
$$= \begin{bmatrix} 2x-y & -x \\ y & x+2y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

At $(x,y) = (1,1)$, $(u,v) = (0,2)$ and the approximating linear mapping is

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

For $dy = 0$, we get $du = dx$ and $dv = dx$, so that the slope $du/dv = 1$. For $dx = 0$, we get $du = -dy$ and $dv = 3\ dy$, so that the slope $du/dv = -\frac{1}{3}$. Thus, for $0 \leq dx, dy \leq 1$, these points correspond to a rectangle of area $\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$. Note also that the sides of the approximating rectangle emanating from $(0,2)$ are tangent to the curved region.

Moreover, at $(1.01, 1.02)$, we have $dx = 0.01$ and $dy = 0.02$. Hence,

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix} = \begin{bmatrix} -0.01 \\ 0.07 \end{bmatrix}$$

so that the approximation is $(u,v) = (-0.01, 2.07)$; the exact value is $(u,v) = (-0.0101, 2.0706)$.

Example 19.4. *Find the Jacobian for*

$$\begin{aligned} y_1 &= x_2^2 + x_3^2 + x_4^2 \\ y_2 &= x_1^2 + x_3^2 + x_4^2 \\ y_3 &= x_1^2 + x_2^2 + x_4^2 \\ y_4 &= x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

at the point $(1,0,0,0)$, *and use it to approximate* $\mathbf{y}$ *at* $(1, 0.1, 0.1, 0.1)$.

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \\ dy_4 \end{bmatrix} = \begin{bmatrix} 0 & 2x_2 & 2x_3 & 2x_4 \\ 2x_1 & 0 & 2x_3 & 2x_4 \\ 2x_1 & 2x_2 & 0 & 2x_4 \\ 2x_1 & 2x_2 & 2x_3 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{bmatrix}$$

At $(1, 0, 0, 0)$ the approximating linear mapping is

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \\ dy_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \\ dx_4 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} dy_1 \\ dy_2 \\ dy_3 \\ dy_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the approximation is unchanged from the function value $(0, 1, 1, 1)$.

HOMEWORK FOR DAY 19. Page 95, #1 parts c through g, #2 parts a, b, and c

HOMEWORK ANSWERS. #1 d) $\begin{bmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \\ 2x & 0 \end{bmatrix}$

#1 e) $\begin{bmatrix} 2xyz & x^2z & x^2y \end{bmatrix}$

#2 a) Jacobian is $\begin{bmatrix} 2x_1 & 2x_2 \\ x_2 & x_1 \end{bmatrix}$; at the point $(2,1)$ and when $dx_1 = 0.04$ and $dx_2 = 0.01$, we have

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0.04 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.18 \\ 0.06 \end{bmatrix}$$

So $\mathbf{f}(2.04, 1.01) \approx (5.18, 2.06)$.

#2 b) Jacobian is $\begin{bmatrix} x_2 & x_1 & -2x_3 \\ x_2 + x_3 & x_1 & x_1 \end{bmatrix}$; at the point $(3,2,1)$ and when $dx_1 = 0.01$, $dx_2 = -0.01$ and $dx_3 = 0.03$, we have

$$\begin{bmatrix} dy_1 \\ dy_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0.01 \\ -0.01 \\ 0.03 \end{bmatrix} = \begin{bmatrix} -0.07 \\ 0.09 \end{bmatrix}$$

So $\mathbf{f}(3.01, 1.99, 1.03) \approx (4.93, 9.09)$.

#2 c) Jacobian is $\begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \\ 2e^x & 0 \end{bmatrix}$; at the point $(0, \pi/2)$ and when $dx = 0.1$ and $dy = 0.03$, we have

$$\begin{bmatrix} du \\ dv \\ dw \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.03 \end{bmatrix} = \begin{bmatrix} -0.03 \\ 0.1 \\ 0.2 \end{bmatrix}$$

So $\mathbf{f}(0.1, 1.6) \approx (-0.03, 1.1, 2.2)$.

20 Section 2.8, Differentials of Composite Functions

Objective. *Students will evaluate differentials of composite functions.*

Theorem 20.1. *If $z = f(x, y)$ and $x = g(t),\ y = h(t)$, then*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}. \tag{20.1}$$

If $z = f(x, y)$ and $x = g(u, v),\ y = h(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Proof. We prove Eq. 20.1. If $x = g(t),\ y = h(t)$, then

$$\Delta x = g(t + \Delta t) - g(t), \qquad \Delta y = h(t + \Delta t) - h(t)$$

so that $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, or, by the Fundamental Lemma,

$$\Delta z = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial x}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y.$$

Hence, upon division of each term by Δt, and then letting $\Delta t \to 0$, we get $\frac{\Delta x}{\Delta t} \to \frac{dx}{dt},\ \frac{\Delta y}{\Delta t} \to \frac{dy}{dt},\ \varepsilon_1 \to 0,\ \varepsilon_2 \to 0$. Thus,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

□

Note that multiplying each term by dt gives $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ as in the Fundamental Lemma.

This is easily generalizable to three or more variables; i.e.,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

This brings us to the remarkable fact that, in order to find partial derivatives, we need only compute differentials, pretending that all variables are functions of some other variable. (See the Theorem on page 98.)

Example 20.1. *Find the first partials of* $z = \dfrac{x^2 - 1}{y}$.

We have $dz = \dfrac{2xy\ dx - (x^2 - 1)\ dy}{y^2}$; thus, $\frac{\partial z}{\partial x} = \frac{2x}{y}$ and $\frac{\partial z}{\partial y} = \frac{1-x^2}{y^2}$.

Example 20.2. *Find the first partials of* $z^2 = x^2 - y^2$.

We have $z\ dz = x\ dx + y\ dy$, so that $\frac{\partial z}{\partial x} = \frac{x}{z}$, $\frac{\partial z}{\partial y} = \frac{y}{z}$, $\frac{\partial x}{\partial z} = \frac{z}{x}$, $\frac{\partial y}{\partial z} = \frac{y}{z}$, $\frac{\partial y}{\partial x} = -\frac{x}{y}$, and $\frac{\partial x}{\partial y} = -\frac{y}{x}$.

Example 20.3. *Find the the value of* dw/dt *when* $t = 0$ *of* $w = xy + z$, *where* $x = \cos t,\ y = \sin t,\ z = t$.

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= -\sin^2 t + \cos^2 t + 1 \\
\left.\frac{dw}{dt}\right|_{t=0} &= 1 + \cos 0 = 2
\end{aligned}$$

Example 20.4. *Find* dy/dx *for* $y = \log_u v$ *where* u *and* v *are functions of* x.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} \\
&= \frac{-\log v}{u\log^2 u}\frac{du}{dx} + \frac{1}{v\log u}\frac{dv}{dx}
\end{aligned}$$

HOMEWORK FOR DAY 20. Page 100, #2, #4, and #10

HOMEWORK ANSWERS. #2

$$\begin{aligned}\frac{dy}{dx} &= \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} \\ &= (vu^{v-1})\frac{du}{dx} + (u^v \log u)\frac{dv}{dx}\end{aligned}$$

#4 We find differentials of the the equations in x, t and y, t to get

$$3x^2\ dx + e^x\ dx - 2t - 1 = 0, \qquad t^2\ dy + 2ty + 2ty\ dy + y^2 - 1 + dy = 0$$

which implies $dx = 1$ and $dy = 1$ when $t = x = y = 0$. Hence,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= e^x \cos y \frac{dx}{dt} - e^x \sin y \frac{dy}{dt} \\ \left.\frac{dz}{dt}\right|_{t=0} &= (e^0 \cos 0)(1) - (e^0 \sin 0)(1) = 1\end{aligned}$$

#10 a) $dz = \dfrac{\cos(x^2y^2 - 1)(2xy^2\ dx + 2x^2y\ dy)}{\sin(x^2y^2 - 1)}$; thus, $\frac{\partial z}{\partial x} = 2xy^2 \cot(x^2y^2 - 1)$ and $\frac{\partial z}{\partial y} = 2x^2y \cot(x^2y^2 - 1)$

#10 c) $2z\ dz = 2x\ dx + 4y\ dy$; thus, $\frac{\partial z}{\partial x} = \frac{x}{z}$ and $\frac{\partial z}{\partial y} = \frac{2y}{z}$.

21 Section 2.9, The General Chain Rule

Objective. *Students will understand and apply the general chain rule to differentiate multivariable composite functions.*

Consider the system

$$\begin{cases} y_1 = f_1(u_1(\mathbf{x}), \ldots, u_p(\mathbf{x})) \\ \vdots \\ y_m = f_m(u_1(\mathbf{x}), \ldots, u_p(\mathbf{x})) \end{cases}$$

Then

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \cdots + \frac{\partial y_i}{\partial u_p}\frac{\partial u_p}{\partial x_j}$$

for $i = 1, \ldots, m,\ j = 1, \ldots, n.$ We can express this using Jacobians as the *General Chain Rule*:

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \left(\frac{\partial y_i}{\partial u_k}\right)\left(\frac{\partial u_k}{\partial x_j}\right)$$

Example 21.1. *Let* $y_1 = u_1u_2 - u_1u_3,\ y_2 = u_1u_3 + u_2^2$ *and let* $u_1 = x_1\cos x_2 + (x_1 - x_2)^2,\ u_2 = x_1\sin x_2 + x_1x_2,\ u_3 = x_1^2 - x_1x_2 + x_2^2$. *Find* $\left(\frac{\partial y_i}{\partial x_j}\right)$, *then evaluate the partial derviatives when* $x_1 = 1,\ x_2 = 0$.

We have

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} u_2 - u_3 & u_1 & -u_1 \\ u_3 & 2u_2 & u_1 \end{bmatrix} \begin{bmatrix} \cos x_2 + 2(x_1 - x_2) & -x_1\sin x_2 - 2(x_1 - x_2) \\ \sin x_2 + x_2 & x_1\cos x_2 + x_1 \\ 2x_1 - x_2 & 2x_2 - x_1 \end{bmatrix}$$

When $x_1 = 1,\ x_2 = 0$, then $u_1 = 2,\ u_2 = 0,\ u_3 = 1.$ Hence

$$\left(\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right)\bigg|_{(1,2)} = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -7 & 8 \\ 7 & -4 \end{bmatrix}$$

Note that the general chain rule can be expressed in terms of differentials:

$$\begin{bmatrix} dy_1 \\ \vdots \\ dy_m \end{bmatrix} = \left(\frac{\partial y_i}{\partial u_k}\right)\left(\frac{\partial u_k}{\partial x_j}\right)\begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

or,

$$\mathbf{y_x} = \mathbf{y_u}\mathbf{u_x}.$$

In the special case where $\mathbf{y}$ and $\mathbf{u}$ are linear, then $\mathbf{y_u} = A$ and $\mathbf{u_x} = B$ are matrices so that $\mathbf{y_x} = AB$.

Example 21.2. *Let $y_1 = u_1u_2 - 3u_1,\ y_2 = u_2^2 + 2u_1u_2 + 2u_1 - u_2$ and let $u_1 = x_1\cos 3x_2,\ u_2 = x_1 \sin x_2$. Find $\left(\frac{\partial y_i}{\partial x_j}\right)$, then evaluate the partial derviatives when $x_1 = x_2 = 0$.*

We have

$$\left(\frac{\partial y_i}{\partial x_j}\right) = \begin{bmatrix} u_2 - 3 & u_1 \\ 2u_2 + 2 & 2u_2 + 2u_1 - 1 \end{bmatrix} \begin{bmatrix} \cos 3x_2 & -3x_1 \sin 3x_2 \\ \sin 3x_2 & 3x_1 \cos 3x_2 \end{bmatrix}$$

When $x_1 = x_2 = 0$, then $u_1 = u_2 = 0$. Hence

$$\left.\left(\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right)\right|_{(0,0)} = \begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 2 & 0 \end{bmatrix}$$

The determinant of this last matrix again corresponds to a volume (or, in this case, area) ratio. *Warning:* $\left(\frac{\partial y_i}{\partial x_j}\right)$ is a matrix, but $\frac{\partial y_i}{\partial x_j}$ is a determinant!

Example 21.3. *Find $\dfrac{\partial(z,w)}{\partial(x,y)}$ for $z = \sqrt{u^2+v^2},\ w = v(u^2+v^2)^{-1/2}$ and $u = (x+y+1)^{-1},\ v = (2x-y+1)^{-1}$ when $x = y = 0$.*

$$\frac{\partial(z,w)}{\partial(x,y)} = \begin{vmatrix} \dfrac{u}{\sqrt{u^2+v^2}} & \dfrac{v}{\sqrt{u^2+v^2}} \\ \dfrac{-uv}{(u^2+v^2)^{3/2}} & \dfrac{u^2}{(u^2+v^2)^{3/2}} \end{vmatrix} \begin{vmatrix} \dfrac{-1}{(x+y+1)^2} & \dfrac{-1}{(x+y+1)^2} \\ \dfrac{-2}{(2x-y+1)^2} & \dfrac{1}{(2x-y+1)^2} \end{vmatrix}$$

$$\left.\frac{\partial(z,w)}{\partial(x,y)}\right|_{(0,0)} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \end{vmatrix} \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = \frac{1}{2}(-3) = -\frac{3}{2}$$

Example 21.4. *Find $\dfrac{\partial(z,w)}{\partial(r,\theta)}$ for $z = x^2 + xy + y^2,\ w = y^3 - 2y + x^2$ where x and y are the polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ at the polar point $(1, \frac{\pi}{2})$.*

$(r, \theta) = (1, \frac{\pi}{2})$ implies $(x, y) = (0, 1)$. Thus,

$$\frac{\partial(z,w)}{\partial(r,\theta)} = \begin{vmatrix} 2x+y & x+2y \\ 2x & 3y^2-2 \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$\left.\frac{\partial(z,w)}{\partial(r,\theta)}\right|_{(1,\frac{\pi}{2})} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = (1)(1) = 1$$

Read problem 6 on page 105.

HOMEWORK FOR DAY 21. Page 104, #1 parts a, b, and c; Page 105, #2 part a, #5

HOMEWORK ANSWERS. #1 b)

$$\left(\frac{\partial(y_1,y_2)}{\partial(x_1,x_2,x_3)}\right)=\begin{bmatrix}2u_1-3 & 2u_2 & 1\\ 2u_1+2 & -2u_2 & -3\end{bmatrix}\begin{bmatrix}x_2x_3^2 & x_1x_3^2 & 2x_1x_2x_3\\ x_2^2x_3 & 2x_1x_2x_3 & x_1x_2^2\\ 2x_1x_2x_3 & x_1x_3^2 & x_1^2x_2\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2 & 1\\ 4 & -2 & -3\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}=\begin{bmatrix}3 & 4 & 1\\ -4 & -3 & 3\end{bmatrix}$$

#1 c)

$$\left(\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2)}\right)=\begin{bmatrix}e^{u_2} & u_1e^{u_2}\\ e^{-u_2} & -u_1e^{-u_2}\\ 2u_1 & 0\end{bmatrix}\begin{bmatrix}2x_1 & 1\\ 4x_1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}e^2 & e^2\\ e^{-2} & e^{-2}\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 4 & -1\end{bmatrix}=\begin{bmatrix}6e^2 & 0\\ -2e^{-2} & 2e^{-2}\\ 4 & 2\end{bmatrix}$$

#2 a)

$$\left(\frac{\partial(z,w)}{\partial(x,y)}\right)$$

$$=\begin{vmatrix}3u^2+6uv+2u & 3u^2-2v^2-2v\\ 3u^2-4u & 3v^2\end{vmatrix}\begin{vmatrix}\cos xy-xy\sin xy & -x^2\sin xy\\ \sin xy+xy\cos xy+2x & x^2\cos xy-2y\end{vmatrix}$$

$$=\begin{vmatrix}11 & -2\\ -1 & 3\end{vmatrix}\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}=\begin{vmatrix}7 & -2\\ 5 & 3\end{vmatrix}=31$$

#5

$$\mathbf{w_x}=\mathbf{w_u u_x}$$

$$=\begin{bmatrix}2 & 11\\ 7 & 5\end{bmatrix}\begin{bmatrix}1+2x_2 & 2x_1-3\\ 2-3x_2 & 5-3x_1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 11\\ 7 & 5\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & -1\end{bmatrix}=\begin{bmatrix}-5 & -9\\ 16 & 2\end{bmatrix}$$

22 Section 2.10, Implicit Functions

Objective. *Students will apply the Implicit Function Theorem to find derivatives of multivariable implicitly-defined functions.*

The differential of an implicit function $F(x, y, z) = 0$ where $z = f(x, y)$ is $F_x dx + F_y dy + F_z dz = 0$. Solving for dz, we have

$$dz = -\frac{F_x}{F_z}\, dx - \frac{F_y}{F_z}\, dy,$$

so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Example 22.1. *Find* $\dfrac{\partial z}{\partial x}$ *and* $\dfrac{\partial z}{\partial y}$ *if* $x^2 + y^2 + z^2 - 1 = 0$.

Answer: $\dfrac{\partial z}{\partial x} = -\frac{x}{z}$ and $\dfrac{\partial z}{\partial y} = -\frac{y}{z}$.

Example 22.2. *Find* $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *if* $x^2z^2 + xy^2 - z^3 + 4yz - 5 = 0$.

Answer: $\dfrac{\partial z}{\partial x} = -\frac{2xz^2+y^2}{2x^2z-3z^2+4y}$ and $\frac{\partial z}{\partial y} = -\frac{2xy+4z}{2x^2z-3z^2+4y}$.

For two implicit functions $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$ where $z = f(x, y),\ w = g(x, y)$ we have the two differential equations

$$F_x dx + F_y dy + F_z dz + F_w dw = 0$$
$$G_x dx + G_y dy + G_z dz + G_w dw = 0$$

This is solved by elimination or Cramer's Rule. By Cramer's Rule we have the formulas involving the Jacobians.

$$\frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} F_x & F_w \\ G_x & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} \qquad \frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} F_y & F_w \\ G_y & G_w \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}} \qquad \frac{\partial w}{\partial y} = -\frac{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_z & F_w \\ G_z & G_w \end{vmatrix}}$$

Example 22.3. *Find all four of the above partials for $2x^2 + y^2 + z^2 - zw = 0$ and $x^2 + y^2 + 2z^2 + zw - 8 = 0$.*

The differentials are

$$\begin{aligned} 4x\;dx + 2y\;dy + (2z - w)\;dz - z\;dw &= 0 \\ 2x\;dx + 2y\;dy + (4z + w)\;dz + z\;dw &= 0 \end{aligned}$$

Thus, using the above formulas, we compute each Jacobian:

$$\frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} 4x & -z \\ 2x & z \end{vmatrix}}{\begin{vmatrix} 2z - w & -z \\ 4z + w & z \end{vmatrix}} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} 2y & -z \\ 2y & z \end{vmatrix}}{\begin{vmatrix} 2z - w & -z \\ 4z + w & z \end{vmatrix}} = -\frac{2y}{3z}$$

$$\frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} 2z - w & 4x \\ 4z + w & 2x \end{vmatrix}}{\begin{vmatrix} 2z - w & -z \\ 4z + w & z \end{vmatrix}} = \frac{x(2z + w)}{z^2}$$

$$\frac{\partial w}{\partial y} = -\frac{\begin{vmatrix} 2z - w & 2y \\ 4z + w & 2y \end{vmatrix}}{\begin{vmatrix} 2z - w & -z \\ 4z + w & z \end{vmatrix}} = \frac{2y(z + w)}{3z^2}$$

Note format of these partials:

$$\frac{\partial \text{dependent}_1}{\partial \text{independent}} = -\frac{\begin{vmatrix} \text{independent} & \text{dependent}_2 \\ \text{independent} & \text{dependent}_2 \end{vmatrix}}{\begin{vmatrix} \text{dependent}_1 & \text{dependent}_2 \\ \text{dependent}_1 & \text{dependent}_2 \end{vmatrix}}$$

We may have any number of dependent variables and equations, but the system is only solvable if the number of dependent variables equals the number of equations!

Also, we assume the equations define functions; they may not! (See bottom of page 111.) If the determinant in the denominator is not 0, then the implicit equations define functions as usual. This is contained in the following.

Theorem 22.1 (Implicit Function Theorem). *Let $F_i(x_p, y_q)$ be implicitly defined functions each with p independent variables and q dependent variables. Let each F_i be defined in a neighborhood of point P_0 and have continuous first-order partial derivatives in this neighborhood. Let each equation $F_i = 0$ be satistifed at P_0 and let $\frac{\partial F_i}{\partial y_p} \neq 0$ at P_0. Then in an appropriate neighborhood of the x-coordinates of P_0, there is a unique set of continuous functions $y_i = f_i(x_p)$ such that these functions give the y-coordinates of P_0 and that $F_i(x_p, f_i(x_p)) = 0$ in the neighborhood. Furthermore, the f_i have continuous partial derivatives.*

(See first paragraph of Section 2.11)

Example 22.4. *Find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ at $x = y = 0$ for $e^u + xu - yv - 1 = 0$ and $e^v - xv + yu - 2 = 0$.*

Note that when $x = y = 0$, then $u = 0,\ v = \log 2$. The differentials are

$$\begin{aligned} u\,dx - v\,dy + (e^u + x)\,du - y\,dv &= 0 \\ -v\,dx + u\,dy + y\,du + (e^v - x)\,dv &= 0 \end{aligned}$$

Hence,

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} u & -y \\ -v & e^v - x \end{vmatrix}}{\begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix}} \qquad \left.\frac{\partial u}{\partial x}\right|_{(0,0)} = -\frac{\begin{vmatrix} 0 & 0 \\ -\log 2 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}} = 0$$

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} -v & -y \\ u & e^v - x \end{vmatrix}}{\begin{vmatrix} e^u + x & -y \\ y & e^v - x \end{vmatrix}} \qquad \left.\frac{\partial u}{\partial y}\right|_{(0,0)} = -\frac{\begin{vmatrix} -\log 2 & 0 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}} = \log 2$$

Example 22.5. *Find $\left(\dfrac{\partial x}{\partial y}\right)_z$, $\left(\dfrac{\partial y}{\partial x}\right)_u$, $\left(\dfrac{\partial z}{\partial u}\right)_x$, $\left(\dfrac{\partial y}{\partial z}\right)_x$ for $2x + y - 3z - 2u = 0$ and $x + 2y + z + u = 0$.*

The differentials are

$$\begin{aligned} 2\ dx + dy - 3\ dz - 2\ du &= 0 \\ dx + 2\ dy + dz + du &= 0 \end{aligned}$$

Hence,

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\dfrac{\partial(F,G)}{\partial(y,u)}}{\dfrac{\partial(F,G)}{\partial(x,u)}} = -\frac{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}} = -\frac{5}{4}$$

$$\left(\frac{\partial y}{\partial x}\right)_u = -\frac{\dfrac{\partial(F,G)}{\partial(x,z)}}{\dfrac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix}} = -\frac{5}{7}$$

$$\left(\frac{\partial z}{\partial u}\right)_x = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(z,y)}} = -\frac{\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}} = -\frac{5}{7}$$

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\dfrac{\partial(F,G)}{\partial(z,u)}}{\dfrac{\partial(F,G)}{\partial(y,u)}} = -\frac{\begin{vmatrix} -3 & -2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}} = \frac{1}{5}$$

Example 22.6. *Page 117, #4*

Part a At the point, the differentials are

$$\begin{aligned} 2\ dx + 2\ dy + 4\ dz - 6\ du + 4\ dv &= 0 \\ 2\ dx - 2\ dy + 4\ dz + 6\ du + 8\ dv &= 0 \end{aligned}$$

Thus, adding equations, we get $4\ dx + 8\ dz + 12\ dv = 0$, or $dv = -\frac{1}{3}(dx + 2\ dz)$. Substituting this into the second differential equation and multiplying each term by 3 gives

$$\begin{aligned} 6\ dx - 6\ dy + 12\ dz + 18\ du - 8\ dx - 16\ dz &= 0 \\ -dx - 3\ dy - 2\ dz + 9\ du &= 0 \\ du &= \tfrac{1}{9}(dx + 3\ dy + 2\ dz) \end{aligned}$$

Part b

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} 2 & 4 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} -6 & 4 \\ 6 & 8 \end{vmatrix}} = \frac{1}{9}, \qquad \frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} 2 & -6 \\ -2 & 6 \end{vmatrix}}{\begin{vmatrix} -6 & 4 \\ 6 & 8 \end{vmatrix}} = 0$$

c) We have $dx = 0.1,\ dy = 0.2,\ dz = -0.2$. Then $du = \frac{1}{9}(0.1 + 3(0.2) - 2(0.2)) = 0.033$ and $dv = -\frac{1}{3}(0.1 - 2(0.2)) = 0.1$. Therefore, $u = 3 + 0.033 = 3.033$ and $v = 2 + 0.1 = 2.1$.

HOMEWORK FOR DAY 22. Page 116, #1, #3 parts a and c (Page 117, #7 is extra credit)

HOMEWORK ANSWERS. #1 a) The differentials are $4x\ dx + 2y\ dy - 2z\ dz = 0$, so $\dfrac{\partial z}{\partial x} = \dfrac{2x}{z}$ and $\dfrac{\partial z}{\partial y} = \dfrac{y}{z}$.

#1 d) The differentials are $ze^{xz}\ dx + ze^{yz}\ dy + \left(ze^{xz} + ye^{yz} + 1\right)\ dz = 0$, so that $\dfrac{\partial z}{\partial x} = \dfrac{ze^{xz}}{ze^{xz} + ye^{yz} + 1}$ and $\dfrac{\partial z}{\partial y} = \dfrac{ze^{yz}}{ze^{xz} + ye^{yz} + 1}$.

#3 a) The differentials are

$$\begin{aligned} 2x\ dx - 2y\ dy + 2u\ du + 4v\ dv &= 0 \\ 2x\ dx + 2y\ dy - 2u\ du - 2v\ dv &= 0 \end{aligned}$$

Hence,

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} 2x & 4v \\ 2x & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{3x}{u}, \qquad \frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} -2y & 4v \\ 2y & -2v \end{vmatrix}}{\begin{vmatrix} 2u & 4v \\ -2u & -2v \end{vmatrix}} = \frac{y}{u}$$

23 Section 2.12, Inverse Functions

Objective. *Students will find derivatives of the inverse of function by differentiating implicitly. Students will interpret polar, cylindrical, and spherical functions as various inverses of rectangular functions.*

The functions $x = f(u, v),\ y = g(u, v)$ constitute a mapping from D_{uv} to D_{xy}. Assume this mapping is one-to-one. Then we have the inverse mapping $u = \varphi(x, y),\ v = \psi(x, y)$. Solving x and y explicitly may be impossible, so to find differentials, we use $f(u, v) - x = 0,\ g(u, v) - y = 0$ and apply implicit differentiation.

Let $F(x, y, u, v) = f(u, v) - x$ and $G(x, y, u, v) = g(u, v) - y$. Then

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -1 & f_v \\ 0 & g_v \end{vmatrix}}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}} = \frac{g_v}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}.$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{f_v}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}, \quad \frac{\partial v}{\partial x} = \frac{-g_u}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}, \quad \frac{\partial v}{\partial y} = \frac{-f_u}{\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}}.$$

Since f and g are inverses with φ and ψ, their Jacobians are inverses as well; i.e., $\dfrac{\partial(x, y)}{\partial(u, v)} \dfrac{\partial(u, v)}{\partial(x, y)} = |I| = 1$; so $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}}$.

Example 23.1. *Find* $\frac{\partial r}{\partial x}$ *and* $\frac{\partial \theta}{\partial x}$ *for the polar relation* $x = r\cos\theta,\ y = r\sin\theta$.

Let $F(x, y, r, \theta) = r\cos\theta - x,\ G(x, y, r, \theta) = r\sin\theta - y$. Then

$$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}} = \frac{r\cos\theta}{r\cos^2\theta + r\sin^2\theta} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}} = \frac{-\sin\theta}{r\cos^2\theta + r\sin^2\theta} = \frac{-\sin\theta}{r}$$

Other coordinate systems are: cylindrical ($x = r\cos\theta,\ y = r\sin\theta,\ z = z$) and spherical ($x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$, where θ is the angle between x and r and where ϕ is the angle between ρ and z.)

HOMEWORK FOR DAY 23. Page 121, #1 parts a and b, #2, #3

HOMEWORK ANSWERS. #1 a) This is easy. b) dr is easy; here is $d\theta$:

$$\begin{aligned} d\theta &= \frac{-y/x^2}{1+y^2/x^2}dx + \frac{1/x}{1+y^2/x^2}dy = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \\ &= \frac{-r\sin\theta}{r^2}dx + \frac{r\cos\theta}{r^2}dy = \frac{-\sin\theta}{r}dx + \frac{\cos\theta}{r}dy \end{aligned}$$

#2 a) We solve the system $u - 2v = x,\ 2u + v = y$ for u and v to get $u = \frac{1}{5}(x+2y)$ and $v = \frac{1}{5}(y-2x)$. b) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5$. The inverse Jacobian is then $\frac{1}{5}$.

#3 a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2$. b) $\left(\dfrac{\partial u}{\partial x}\right)_y = \dfrac{2u}{4u^2+4v^2} = \dfrac{u}{2u^2+2v^2}$, $\left(\dfrac{\partial v}{\partial x}\right)_y = \dfrac{-v}{2u^2+v^2}$.

24 Section 2.13, Geometrical Applications

Objective. *Students will find tangent lines to curves in space and tangent planes to surfaces in space at a point. Students will define and use the gradient of a vector.*

We extend the two-dimensional vector-valued function doncept to three-dimensions. If a curve has $x = f(t),\ y = g(t),\ z = h(t)$, then this is the same as the vector-valued function $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with velocity vector $\mathbf{v} = \mathbf{r}'(t) = x'\mathbf{i} + y'\mathbf{j} + z'\mathbf{k}$ which is tangent to the curve and has magnitude $||\mathbf{v}|| = ds/dt$ where s is the distance along the curve. The unit tangent vector is $\mathbf{T} = \mathbf{v}/||\mathbf{v}||$.

As long as the value $t = t_1$ does not give $\mathbf{v} = \mathbf{0}$, we have

$$\begin{aligned} d\mathbf{r} &= \langle dx,\ dy,\ dz\rangle \\ &= \langle f'(t)dt,\ g'(t)dt,\ h'(t)dt\rangle \end{aligned}$$

and, interpreting dt as the change $t - t_1$, this becomes

$$= \langle f'(t)(t-t_1),\ g'(t)(t-t_1),\ h'(t)(t-t_1)\rangle$$

But from the first equation to the last, the vector components must be equal; hence we have a parametric equation of the *tangent line*:

$$x - x_1 = f'(t)(t-t_1), \quad y - y_1 = g'(t)(t-t_1), \quad z - z_1 = h'(t)(t-t_1)$$

Example 24.1. *Find tangent line to the curve $x = \sin t,\ y = \cos t,\ z = \sin^2 t$ at $t = \pi/3$.*

When $t = \pi/3$, we have $x = \frac{\sqrt{3}}{2},\ y = \frac{1}{2},\ z = \frac{3}{4}$. Also, $x'(\frac{\pi}{3}) = \cos\frac{\pi}{3} = \frac{1}{2},\ y'(\frac{\pi}{3}) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2},\ z'(\frac{\pi}{3}) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. Hence, the tangent line is $x - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(t - \frac{\pi}{3}\right),\ y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(t - \frac{\pi}{3}\right),\ z - \frac{3}{4} = \frac{\sqrt{3}}{2}\left(t - \frac{\pi}{3}\right)$.

If $F(x, y, z) = 0$ is a surface with curve $x = f(t),\ y = g(t),\ z = h(t)$ in its surface, then $F(f(t), g(t), h(t)) = 0$. Thus,

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z} = 0 \tag{24.1}$$

so that the *tangent plane to the surface containing the tangent line to the curve* at $t = t_1$ is

$$\left.\frac{\partial F}{\partial x}\right|_{t_1}(x - x_1) + \left.\frac{\partial F}{\partial y}\right|_{t_1}(y - y_1) + \left.\frac{\partial F}{\partial z}\right|_{t_1}(z - z_1) = 0 \tag{24.2}$$

Note that this no longer depends on the curve. —[[STEWART 43]]—

Example 24.2. *Find the tangent plane to the surface* $z = 2x^2 + y^2$ *at* $(1, 1, 3)$.

Set equal to zero; then the partials are $4x$, $2y$, and -1; the plane is then $4(x-1)+2(y-1)-(z-3) = 0$, or $4x+2y+z = 3$.

Example 24.3. *Find the tangent plane to the surface* $x^2 + 2y^2 + z = 2$ *at* $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{3}{4}\right)$.

The partials are $2x$, $4y$, and 1; the plane is then $\sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right) + 2\left(y - \frac{1}{2}\right) + z - \frac{3}{4} = 0$, or $\sqrt{3}x + 2y + z = \frac{13}{4}$. Note that this surface contains the curve from the previous example; so this is the plane containing the tangent line from the previous example. —[[STEWART 45]]—

Example 24.4. *Find the tangent plane to* $x^2 + y^2 + z^2 = 14$ *at* $(1, 2, 3)$. *What is the normal vector to the plane?*

The partials are $2x$, $2y$, and $2z$; the plane is $2(x-1)+4(y-2)+6(z-3) = 0$, or $2x+4y+6z = 28$. The normal is the vector $2\mathbf{i}+4\mathbf{j}+6\mathbf{k}$.

Note that Eq. 24.1 implies that $\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle$ is normal to the surface F. This is called the *gradient vector of* F, denoted ∇F or grad F. Thus Eq. 24.2 can be written as $\nabla F \cdot d\mathbf{r} = 0$. —[[LARSON 103]]—

If the curve is defined by the intersection of two surfaces F and G, then the intersection of the tangent planes $\nabla F \cdot d\mathbf{r} = 0$ and $\nabla G \cdot d\mathbf{r} = 0$ is the tangent line to the curve. Note that since $d\mathbf{r}$ is perpendicular to both gradients, then $d\mathbf{r} \times (\nabla F \times \nabla G) = \mathbf{0}$.

Example 24.5. *Find the tangent line to the curve defined by the intersection of* $F : 2x + y - z = 6$ *and* $G : z + 2y + 2z = 7$ *at* $(3, 1, 1)$.

The tangent plane to F is

$$\begin{aligned}
\nabla F \cdot d\mathbf{r}\big|_{t_1} &= 0 \\
\langle 2, 1, -1\rangle \cdot \langle dx, dy, dz\rangle\big|_{(3,1,1)} &= 0 \\
\langle 2, 1, -1\rangle \cdot \langle x-3, y-1, z-1\rangle &= 0 \\
2(x-3) + y - 1 - (z-1) &= 0 \\
2x + y - z &= 6.
\end{aligned}$$

The tangent plane to G is

$$\begin{aligned}
\nabla G \cdot d\mathbf{r}|_{t_1} &= 0 \\
\langle 1,2,2\rangle \cdot \langle dx,dy,dz\rangle|_{(3,1,1)} &= 0 \\
\langle 1,2,2\rangle \cdot \langle x-3,y-1,z-1\rangle &= 0 \\
x-3+2(y-1)+2(z-1) &= 0 \\
x+2y+2z &= 7.
\end{aligned}$$

The intersection of the two planes is then $x = 3+4t,\ y = 1-5t,\ z = 1+3t$.

Example 24.6. *Find the line tangent to the curve defined by the intersection of $F : x^2+y^2+z^2 = 9$ and $G : z^2+y^2-8z^2 = 0$ at $(2,2,1)$.*

The tangent plane to F is

$$\begin{aligned}
\nabla F \cdot d\mathbf{r}|_{t_1} &= 0 \\
\langle 2x,2y,2x\rangle \cdot \langle dx,dy,dz\rangle|_{(2,2,1)} &= 0 \\
\langle 4,4,2\rangle \cdot \langle x-2,y-2,z-1\rangle &= 0 \\
4(x-2)+4(y-2)+2(z-1) &= 0 \\
2x+2y+z &= 9.
\end{aligned}$$

The tangent plane to G is

$$\begin{aligned}
\nabla G \cdot d\mathbf{r}|_{t_1} &= 0 \\
\langle 2x,2y,-16x\rangle \cdot \langle dx,dy,dz\rangle|_{(2,2,1)} &= 0 \\
\langle 4,4,-16\rangle \cdot \langle x-2,y-2,z-1\rangle &= 0 \\
4(x-2)+4(y-2)-16(z-1) &= 0 \\
x+y-4z &= 0.
\end{aligned}$$

The intersection of the two planes is then $x = 2+t,\ y = 2-t,\ z = 1$.

Example 24.7. *Find the tangent plane and the normal line to $z = x/y$ at $(2,1,2)$.*

The tangent plane is

$$\begin{aligned} \nabla f \cdot d\mathbf{r}\big|_{t_1} &= 0 \\ \langle 1/y, -x/y^2, 1\rangle \cdot \langle dx, dy, dz\rangle\Big|_{(2,2,1)} &= 0 \\ \langle 1, -2, 1\rangle \cdot \langle x-2, y-1, z-2\rangle &= 0 \\ x - 2 - 2(y-1) + (z-2) &= 0 \\ x - 2y + z &= 2. \end{aligned}$$

The normal line is $x = 2 + t,\ y = 1 - 2t,\ z = 2 + t.$

HOMEWORK FOR DAY 24. Page 128, #8 parts a, b, and c; Page 129, #10 parts a and d, #11 part c, #15 part a

HOMEWORK ANSWERS. #8 a) From Example 24.6, the tangent plane is $2x+2y+z=9$. The normal is $x=2+2t,\ y=2+2t,\ z=1+t$.

#8 b) The tangent plane is

$$\begin{aligned}
\nabla f \cdot d\mathbf{r}\big|_{t_1} &= 0 \\
\langle 2xe^{x^2+y^2}, 2ye^{x^2+y^2}, -2z\rangle \cdot \langle dx, dy, dz\rangle\Big|_{(0,0,1)} &= 0 \\
\langle 0, 0, -2\rangle \cdot \langle x, y, z-1\rangle &= 0 \\
-2(z-1) &= 0 \\
z &= 1.
\end{aligned}$$

The normal line is the z-axis.

#8 c) The tangent plane is

$$\begin{aligned}
\nabla f \cdot d\mathbf{r}\big|_{t_1} &= 0 \\
\langle 3x^2-y^2, -2xy+z^2, 2yz-3z^2\rangle \cdot \langle dx, dy, dz\rangle\big|_{(1,1,1)} &= 0 \\
\langle 2, -1, -1\rangle \cdot \langle x-1, y-1, z-1\rangle &= 0 \\
2(x-1)-(y-1)-(z-1) &= 0 \\
2x-y-z &= 0.
\end{aligned}$$

The normal line is $x=1+2t,\ y=1-t,\ z=1-t$.

#10 a) The tangent plane is

$$\begin{aligned}
\nabla f \cdot d\mathbf{r}\big|_{t_1} &= 0 \\
\langle 2x, 2y, 1\rangle \cdot \langle dx, dy, dz\rangle\big|_{(1,1,2)} &= 0 \\
\langle 2, 2, -1\rangle \cdot \langle x-1, y-1, z-2\rangle &= 0 \\
2(x-1)+2(y-1)-(z-2) &= 0 \\
2x+2y+z &= 6.
\end{aligned}$$

The normal line is $x=1+2t,\ y=1+2t,\ z=2+t$.

#11 c) The tangent plane to F is

$$\begin{aligned}
\nabla F \cdot d\mathbf{r}\big|_{t_1} &= 0 \\
\langle 2x, 2y, 0\rangle \cdot \langle dx, dy, dz\rangle\big|_{(1,0,-1)} &= 0 \\
\langle 2, 0, 0\rangle \cdot \langle x-1, y, z+1\rangle &= 0 \\
2(x-1) &= 0 \\
x &= 1.
\end{aligned}$$

The tangent plane to G is the plane itself, $x+y+z=0$. The intersection of the planes is $x=1,\ y=-1+t,\ z=-t$.

#15 a) The surface is a sphere centered at the origin of radius $r = \sqrt{x^2+y^2+z^2}$; since $\nabla F = \langle 2x, 2y, 2z\rangle$, this represents a vector of length $||\nabla F|| = 2\sqrt{x^2+y^2+z^2} = 2r$ in the same direction as the radius. Since all radii are perpendicular to a sphere, ∇F must be normal to F.

25 Section 2.14, The Directional Derivative

Objective. *Students will compute the directional derivative of a function.*

Consider the ratio of change in F to the distance moved in a given direction, say $\mathbf{v}$. This is called the *directional derivative of* F, denoted $\nabla_{\mathbf{v}} F$. —[[Larson 101]]—

A movement from (x, y, z) in direction $\mathbf{v}$ corresponds to increments proportional to components of $\mathbf{v}$; i.e., $\Delta x = hv_x,\ \Delta y = hv_y,\ \Delta z = hv_z$ for scalar h. Thus, the displacement is $h\mathbf{v}$ and its magnitude is $h||\mathbf{v}||$. Hence,

$$\frac{\Delta F}{h||\mathbf{v}||} = \frac{\partial F}{\partial x}\frac{v_x}{||\mathbf{v}||} + \frac{\partial F}{\partial y}\frac{v_y}{||\mathbf{v}||} + \frac{\partial F}{\partial z}\frac{v_z}{||\mathbf{v}||} + \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

As $h \to 0$, $\varepsilon_{1,2,3} \to 0$ and $\frac{\Delta F}{h||\mathbf{v}||} \to \nabla_{\mathbf{v}} F$. So,

$$\nabla_{\mathbf{v}} F = \frac{\partial F}{\partial x}\frac{v_x}{||\mathbf{v}||} + \frac{\partial F}{\partial y}\frac{v_y}{||\mathbf{v}||} + \frac{\partial F}{\partial z}\frac{v_z}{||\mathbf{v}||}$$

But each $v_i/||\mathbf{v}||$ is a direction cosine of $\mathbf{v}$; thus, we can rewrite this as

$$\nabla_{\mathbf{v}} F = \frac{\partial F}{\partial x}\cos\alpha + \frac{\partial F}{\partial y}\cos\beta + \frac{\partial F}{\partial z}\cos\gamma$$

Each partial alone is the directional derivative in the respective axial direction.

If $\mathbf{v} = \nabla F$ (i.e., if the direction is the gradient) then $\nabla_{\mathbf{v}} F$ attains its maximum value with maximum given by

$$||\nabla_{\mathbf{v}} F|| = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}$$

The gradient points in the direction in which F increases most rapidly, and its magnitude is the rate of increase in that direction.

If θ is the angle between $\mathbf{v}$ and ∇F, then $\nabla_{\mathbf{v}} F = ||\nabla f|| \cos\theta$. Hence, if $\mathbf{v}$ is tangent to the surface, then $\nabla_{\mathbf{v}} F = 0$.

(The gradient is the steepest ascent.) —[[Larson 102]]—

Example 25.1. *Find the directional derivative of a)* $F = x^2 y$ *at* $(\sqrt{2}, 1)$ *and b)* $G = e^{xy}$ *at* $(\sqrt{2}, 0)$ *in the directions of* $\mathbf{i} + \mathbf{j}$ *and* $\mathbf{i} - \mathbf{j}$.

Part a $\nabla_{\mathbf{v}}F = (2xy)(\cos\frac{\pi}{4}) + (x^2)(\sin\frac{\pi}{4}) = xy\sqrt{2} + \frac{x^2}{\sqrt{2}}$; at the point, this becomes $2+\sqrt{2}$. $\nabla_{\mathbf{v}}F = (2xy)(\cos\frac{-\pi}{4}) + (x^2)(\sin\frac{-\pi}{4}) = xy\sqrt{2} - \frac{x^2}{\sqrt{2}}$; at the point, this becomes $2-\sqrt{2}$.

Part b $\nabla_{\mathbf{v}}G = (ye^{xy})(\frac{1}{\sqrt{2}}) + (xe^{xy})(\frac{1}{\sqrt{2}}) = \frac{e^{xy}}{\sqrt{2}}(y+x)$; at the point, this becomes 1. $\nabla_{\mathbf{v}}G = (ye^{xy})(\frac{1}{\sqrt{2}}) - (xe^{xy})(\frac{1}{\sqrt{2}}) = \frac{e^{xy}}{\sqrt{2}}(y-x)$; at the point, this becomes -1.

Example 25.2. *Find the directional derivative of* $F = xe^y + \cos xy$ *at the point* $(2,0)$ *in the direction of* $\mathbf{v} = \langle 3, -4\rangle$.

We have

$$\begin{aligned}\nabla_{\mathbf{v}}F\big|_{(2,0)} &= \frac{\partial F}{\partial x}\frac{v_x}{||\mathbf{v}||} + \frac{\partial F}{\partial y}\frac{v_y}{||\mathbf{v}||}\bigg|_{(2,0)} \\ &= (e^y - y\sin xy)(\tfrac{3}{5}) + (xe^y - x\sin xy)(-\tfrac{4}{5})\big|_{(2,0)} \\ &= (1)(\tfrac{3}{5}) + (2)(-\tfrac{4}{5}) = -1\end{aligned}$$

Example 25.3. *Find the directional derivative of* $w = F(x,y,z) = z^2e^{xy^2}$ *at the point* $(1,0,\sqrt{3})$ *in the direction a)* $\langle -1,-1,-1\rangle$*; b)* $\langle 0,0,-1\rangle$.

The partials are $-y^2z^2e^{xy^2}$, $-2xyz^2e^{xy^2}$, and $-2ze^{xy^2}$. Thus the directional derivatives are a) $\nabla_{\mathbf{v}}F = (0)(-\frac{1}{\sqrt{3}}) + (0)(-\frac{1}{\sqrt{3}}) - 2\sqrt{3}(-\frac{1}{\sqrt{3}}) = 2$ and b) $\nabla_{\mathbf{v}}F = (0)(0) + (0)(0) - 2\sqrt{3}(-1) = 2\sqrt{3}$.

Example 25.4. *Find the directional derivative of* $F = 3x - 5y + 2z$ *at* $(2,2,1)$ *in the direction of the outer normal of the surface* $x^2+y^2+z^2 = 9$.

From Homework 2.13, #8a, the normal line is $x = 2+2t,\ y = 2+2t,\ z = 1+t$, so a vector in that direction is for any $t > 0$; choose $t = 1$ to get $\mathbf{v} = \langle 4,4,2\rangle$. Thus, $||\mathbf{v}|| = 6$. Therefore,

$$\nabla_{\mathbf{v}}F = 3(\tfrac{2}{3}) - 5(\tfrac{2}{3}) + 2(\tfrac{1}{3}) = -\tfrac{2}{3}$$

Example 25.5. *Page 134, #1 part f*

The two surfaces are an elliptical cone and a hyperboloid of one sheet; the curve of intersection is an ellipse defined parametrically by $\mathbf{v}$ =

$\langle 5\cos t, 5\sin t, 5\rangle$ with $||\mathbf{v}|| = 5\sqrt{2}$. Since the point is $(3,4,5)$, we have t defined by $\arccos\frac{3}{5} = \arcsin\frac{4}{5}$. Hence,

$$\begin{aligned}
\nabla_{\mathbf{v}} F|_{(3,4,5)} &= 2x\frac{5\cos t}{5\sqrt{2}} + 2y\frac{5\sin t}{5\sqrt{2}} - 2z\frac{5}{5\sqrt{2}}\bigg|_{(3,4,5)} \\
&= (6)(\tfrac{3}{5\sqrt{2}}) + (8)(\tfrac{4}{5\sqrt{2}}) - (10)\frac{1}{\sqrt{2}} \\
&= \frac{50}{5\sqrt{2}} - \frac{10}{\sqrt{2}} = 0
\end{aligned}$$

HOMEWORK FOR DAY 25. Page 134, #1 parts a, b, c, and d

Homework Answers. #1

a) Direction vector is $\mathbf{v} = \langle 2, 3, -3 \rangle$, so $||\mathbf{v}|| = \sqrt{22}$. The gradient is $\nabla F = \langle 4x, -2y, 2z \rangle$; at the point, $\nabla F = \langle 4, -4, 6 \rangle$. Hence, $\nabla_{\mathbf{v}} F = \langle 4, -4, 6 \rangle \cdot \langle 2, 3, -3 \rangle / \sqrt{22} = (8 - 12 - 18)/\sqrt{22} = -\sqrt{22}$.

b) Direction vector is $\mathbf{v} = \langle a, b, c \rangle$, so $||\mathbf{v}|| = \sqrt{a^2 + b^2 + c^2}$. The gradient is $\nabla F = \langle 2x, 2y, 0 \rangle$; at the point, $\nabla F = \langle 0, 0, 0 \rangle$. Hence, $\nabla_{\mathbf{v}} F = 0$; and every vector $\mathbf{v}$ is perpendicular to ∇F at the origin.

c) Direction vector is $\mathbf{v} = \langle \cos 60°, \sin 60° \rangle = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$; note this is a unit vector. The gradient is $\nabla F = \langle e^x \cos y, -e^x \sin y \rangle$; at the point, $\nabla F = \langle 1, 0 \rangle$. Hence, $\nabla_{\mathbf{v}} F = \langle 1, 0 \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle = \frac{1}{2}$.

26 Section 2.15, Higher Order Partial Derivatives

Objective. *Students will compute all second partial derivatives. Students will prove that both mixed partial derivatives of a function are equal. Students will apply the Laplacian of a function to determine whether a given function is harmonic.*

We begin with the case where $z = F(x, y)$ is a differentiable function on domain D. First partials are themselves functions, so they can be further differentiated on D.

Example 26.1. *Find all second partial derivatives of* $z = F(x, y) = x^3 \cos y$.

We have $\frac{\partial z}{\partial x} = 3x^2 \cos y$ and $\frac{\partial z}{\partial y} = -x^2 \sin y$. Thus,

$$F_{xx} = \frac{\partial^2 z}{\partial x^2} = 6x \cos y \qquad F_{yy} = \frac{\partial^2 z}{\partial y^2} = -z^2 \cos y$$

$$F_{yx} = \frac{\partial^2 z}{\partial x \partial y} = -3x^2 \sin y \qquad F_{xy} = \frac{\partial^2 z}{\partial y \partial x} = -3x^2 \sin y$$

It is not coincidence that the *"mixed partials"* are the same.

Theorem 26.1 (Clairaut's Mixed Partial Derivative Theorem). *Let $F(x, y)$ be continuous in D so that F_x, F_y, F_{yx}, F_{xy} are also continuous. Let $(a, b) \in D$. Then $F_{yx}(a, b) = F_{xy}(a, b)$.*

Proof. Choose $h \neq 0, k \neq 0$ and $\delta > 0$ such that $(a + h, b + k)$ is in a neighborhood of (a, b) of radius δ. Consider the following function.

$$\varphi(h, k) = [F(a + h, b + k) - F(a + h, b)] - [F(a, b + k) - F(a, b)]$$

Use the Mean Value Theorem in the variable a on the function $G(a) = F(a, b + k) - F(a, b)$ on $[a, a + h]$. We then have $G(a + h) - G(a) = hG'(\alpha_1)$ where $a < \alpha_1 < a + h$. In terms of φ and F, this implies

$$\varphi(h, k) = h\,[F_x(\alpha_1, b + k) - F_x(\alpha_1, b)]$$

Now, apply the Mean Value Theorem again to the function $H(b) = F_x(\alpha_1, b)$ on $[b, b + k]$. Thus,

$$\varphi(h, k) = hkF_{xy}(\alpha_1, \beta_1)$$

where $b < \beta_1 < b + k$. Next, we rewrite φ as

$$\varphi(h,k) = [F(a+h,b+k) - F(a,b+k)] - [F(a+h,b) - F(a,b)]$$

and use the same procedure as before, but on the variable b. We obtain

$$\varphi(h,k) = hkF_{yx}(\alpha_2,\beta_2),$$

where $a < \alpha_2 < a+h$ and $b < \beta_2 < b+k$. Hence, $F_{xy}(\alpha_1,\beta_1) = F_{yx}(\alpha_2,\beta_2)$. Now let $h \to 0$, $k \to 0$. Then $\alpha_1 \to a$, $\alpha_2 \to a$, $\beta_1 \to b$, $\beta_2 \to b$, and $F_{yx}(a,b) = F_{xy}(a,b)$. □

Example 26.2. *Find all second order partials of* $z = F(x,y) = x^2 + xy + y^2$.

$\frac{\partial z}{\partial x} = 2x + y$, $\frac{\partial z}{\partial y} = x + 2y$, $\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial y^2} = 2$, $\frac{\partial^2 z}{\partial x \partial y} = 1$.

Example 26.3. *Find all second order partials of* $z = F(x,y) = x\tan y$ *at the point* $(3, \frac{\pi}{4})$.

$\frac{\partial z}{\partial x} = \tan y$, $\frac{\partial z}{\partial y} = x\sec^2 y$, $\frac{\partial^2 z}{\partial x^2} = \sec^2 y$, $\frac{\partial^2 z}{\partial y^2} = 2x\sec^2 y\tan y$, $\frac{\partial^2 z}{\partial x \partial y} = \sec^2 y$. At the point, these become: $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 6$, $\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial y^2} = 12$, $\frac{\partial^2 z}{\partial x \partial y} = 2$.

Example 26.4. *Find all second order partials of* $z = F(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$.

We have $\frac{\partial z}{\partial x} = -x(x^2+y^2)^{-3/2}$, $\frac{\partial z}{\partial y} = -y(x^2+y^2)^{-3/2}$. Thus,

$$\frac{\partial^2 z}{\partial x^2} = -(x^2+y^2)^{-3/2} + 3x^2(x^2+y^2)^{-5/2} = \frac{2x^2-y^2}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial^2 z}{\partial y^2} = -(x^2+y^2)^{-3/2} + 3y^2(x^2+y^2)^{-5/2} = \frac{2y^2-x^2}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 3xy(x^2+y^2)^{-5/2} = \frac{3xy}{(x^2+y^2)^{5/2}}$$

Example 26.5. *Find all second order partials of* $z = \arctan \frac{y}{x}$.

We have $\frac{\partial z}{\partial x} = -y/(x^2+y^2)$, $\frac{\partial z}{\partial y} = x/(x^2+y^2)$. Thus,

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Third-order partials: $\dfrac{\partial^3 F}{\partial x^2}, \dfrac{\partial^3 F}{\partial y^3}, \dfrac{\partial^3 F}{\partial x^2 \partial y}, \dfrac{\partial^3 F}{\partial x \partial y^2}$.

The *Laplacian* of F, denoted $\nabla^2 F$, is the quantity $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$. If $w = F(x, y, z)$, then $\nabla^2 F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}$. (These are called Laplace's equations.) Note that $\nabla^2 = \nabla \cdot \nabla$.

If $\nabla^2 F = 0$, then F is called *harmonic*. Used in electromagnetic fields, fluid dynamics, heat conduction.

Define $\nabla^2(\nabla^2 F) = \nabla^4 F = \dfrac{\partial^4 F}{\partial x^4} + 2\dfrac{\partial^4 F}{\partial x^2 \partial y^2} + \dfrac{\partial^4 F}{\partial y^4}$. If $\nabla^4 F = 0$ then F is called *biharmonic*. Used in elasticity.

Example 26.6. *Show that* $z = \log \sqrt{x^2 + y^2}$ *is harmonic.*

We have $\frac{\partial z}{\partial x} = x/(x^2 + y^2)$, $\frac{\partial z}{\partial y} = y/(x^2 + y^2)$, so the second partials are $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$. Since $\nabla^2 z = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, z is harmonic.

HOMEWORK FOR DAY 26. Page 143, #1 part c, #2 part b, #3 parts a and b, #4 part b (#4 part a is extra credit)

HOMEWORK ANSWERS. #1 c) We have $\frac{\partial w}{\partial x} = 2xe^{x^2-y^2}$ so $\frac{\partial^2 w}{\partial x \partial y} = -4xye^{x^2-y^2}$. Now we differentiate $\frac{\partial^2 w}{\partial x \partial y}$ with respect to x and again with y, to find

$$\frac{\partial^3 w}{\partial x^2 \partial y} = -4ye^{x^2-y^2} - 8x^2ye^{x^2-y^2} = -4ye^{x^2-y^2}(1+2x^2)$$
$$\frac{\partial^3 w}{\partial x \partial y^2} = -4xe^{x^2-y^2} + 8xy^2e^{x^2-y^2} = -4xe^{x^2-y^2}(1-2y^2)$$

#2 b) $\frac{\partial w}{\partial x} = x(x^2+y^2+z^2)^{-1/2}$ so that $\dfrac{\partial^2 w}{\partial x \partial y} = -xy(x^2+y^2+z^2)^{-3/2}$ and $\dfrac{\partial^3 w}{\partial x \partial y \partial z} = 3xyz(x^2+y^2+z^2)^{-5/2}$

#3 a) $\frac{\partial w}{\partial x} = e^x \cos y$, $\frac{\partial w}{\partial y} = -e^x \sin y$. Hence,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = e^x \cos y - e^x \cos y = 0.$$

#3 b) $\frac{\partial w}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial w}{\partial y} = -6xy$. Hence,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 6x - 6x = 0.$$

#4 a)

$$\begin{aligned}
\nabla^2(\nabla^2 F) &= \frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} \\
&= \frac{\partial^2}{\partial x^2}\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2 F}{\partial y^2} \\
&= \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)
\end{aligned}$$

In the parantheses, we have $\nabla^2 F$; but F is harmonic, so $\nabla^2 F = 0$:

$$= \frac{\partial^2}{\partial x^2}(0) + \frac{\partial^2}{\partial y^2}(0) = 0$$

Therefore, F is biharmonic as well.

#4 b) For $F = xe^x \cos y$, the required partial derivatives should be $\frac{\partial^4 F}{\partial x^4} = (x+4)e^x \cos y$, $\frac{\partial^4 F}{\partial y^4} = xe^x \cos y$, $\frac{\partial^4 F}{\partial x^2 \partial y^2} = -(x+2)e^x \cos y$.

For $F = x^2 - 3x^2y^2$, the required partial derivatives should be $\frac{\partial^4 F}{\partial x^4} = 24$, $\frac{\partial^4 F}{\partial y^4} = 0$, $\frac{\partial^4 F}{\partial x^2 \partial y^2} = -12$.

27 Section 2.19, Maxima and Minima of Several Variable Functions

Objective. *Students will find critical points of functions of several variables and determine whether these points are maxima, minima, or saddle points.*

Review of Second Derivative Test from single-variable calculus:

Theorem 27.1 (Derivative Test for Extrema). *Let $f'(x_0) = 0$, $f''(x_0) = 0$, …, $f^{(n-1)}(x_0) = 0$, but $f^{(n)} \neq 0$. Then $f(x)$ has a relative maximum at x_0 if n is even and $f^{(n+1)}(x_0) < 0$; $f(x)$ has a relative minimum at x_0 if n is even and $f^{(n+1)}(x_0) > 0$; $f(x)$ has neither a relative minimum nor relative maximum at x_0 but a horizontal inflection point at x_0 if n is odd.*

All multi-variable extrema are similar to single-variable. We consider two-variable in depth. —[[Larson 105]]—

Defintion Points at which $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ are critical points of z.

Do we determine the nature of critical points by second partials? Yes, but there are complications:

Example 27.1. *If $z = 1 + x^2 - y^2$ then $(0, 0)$ is a critical point, but it has a minimum with respect to x when $y = 0$ and maximum with respect to y when $x = 0$.*

—[[Stewart 46]]—

Example 27.2. *If $z = 1 - x^2 + 4xy - y^2$ then $(0, 0)$ is a critical point, and when $y = 0$ or $x = 0$, there is a maximum with respect to x and y; but when $y = x$, there is a minimum at $x = 0$.*

The critical points in the above examples do not give max/min of the surface–these are *saddle points*. In order to account for these points, we use the directional derivative in a direction α:

$$\nabla_\alpha z = \frac{\partial z}{\partial x} \cos \alpha + \frac{\partial z}{\partial y} \sin \alpha.$$

At a critical point we have $\nabla_\alpha z = 0$. However, the type of critical point may vary with the direction chosen, so we use the second directional derivative in the direction α:

$$\begin{aligned}\nabla_\alpha \nabla_\alpha z &= \nabla_\alpha \left(\frac{\partial z}{\partial x} \cos\alpha + \frac{\partial z}{\partial y} \sin\alpha \right) \\ &= \frac{\partial^2 z}{\partial x^2} \cos^2\alpha + 2\frac{\partial^2 z}{\partial x \partial y} \sin\alpha\cos\alpha + \frac{\partial^2 z}{\partial y^2} \sin^2\alpha\end{aligned}$$

We then determine whether $\nabla_\alpha \nabla_\alpha z$ is positive or negative for all α. Thus, analysis of the extrema of z reduces to the analysis of the expression

$$A\cos^2\alpha + 2B\sin\alpha\cos\alpha + C\sin^2\alpha \tag{27.1}$$

where A, B, and C are the partials above.

Theorem 27.2. *If $B^2 - AC < 0$ and $A + C < 0$, then Eq. 27.1 is negative for all α; if $B^2 - AC < 0$ and $A + C > 0$, the Eq. 27.1 is positive for all α.*

Proof. Denote Eq. 27.1 by $P(\alpha)$. Let $B^2 - AC < 0$ and $A + C < 0$.

Then $P(\pm\pi/2) = C < 0$, for if $C \geq 0$, then $A + C < 0$ implies $A < 0$, so that $AC \leq 0$; this contradicts the fact that $B^2 - AC < 0$.

Also, $P(0) = A < 0$, for if $A \geq 0$, then $A + C < 0$ implies $C < 0$, so that $AC \leq 0$; this again contradicts $B^2 - AC < 0$.

For $\alpha \neq \pm\pi/2$, we factor $P(\alpha)$ to get

$$P(\alpha) = \cos^2\alpha(A + 2B\tan\alpha + C\tan^2\alpha).$$

Thus, $P(\alpha)$ is positive, negative, or zero according to whether the quadratic in $\tan\alpha$ is positive, negative, or zero. Since $B^2 - AC < 0$, the quadratic has no real roots; thus the quadratic is always positive or always negative. For $\tan\alpha = 0$, $A + 2B\tan\alpha + C\tan^2\alpha = A < 0$. Hence, the quadratic, and therefore $P(\alpha)$, is always negative.

The second statement is proved similarly. ☐

Theorem 27.3. *Let $z = f(x, y)$ be defined and have continuous first and second derivatives in a domain D. Let (x_0, y_0) be a point of D for which $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero. Let*

$$A = \frac{\partial^2 z}{\partial x^2}(x_0, y_0), \qquad B = \frac{\partial^2 z}{\partial x \partial y}(x_0, y_0), \qquad C = \frac{\partial^2 z}{\partial y^2}(x_0, y_0).$$

Then we have the following cases:

$B^2 - AC < 0$ *and* $A + C < 0$, *relative maximum at* (x_0, y_0)*;*

$B^2 - AC < 0$ *and* $A + C > 0$, *relative minimum at* (x_0, y_0)*;*

$B^2 - AC > 0$, *saddle point at* (x_0, y_0)*;*

$B^2 - AC = 0$, *inconclusive.*

Example 27.3. *Find critical points and test for extrema on* $z = xy^2 + x^2y - xy$*.*

We have

$$\begin{aligned} \frac{\partial z}{\partial x} &= y^2 + 2xy - y = 0 \\ y(y + 2x - 1) &= 0 \\ y &= 0,\ y = 1 - 2x \end{aligned}$$

Substitute each into $\frac{\partial z}{\partial y} = 2xy + x^2 - x = 0$ and solve. When $y = 0$, $x^2 - x = 0$ implies $x = 0$ or $x = 1$. When $y = 1 - 2x$, we have

$$\begin{aligned} 2x(1 - 2x) + x^2 - x &= 0 \\ x(1 - 3x) &= 0 \\ x &= 0,\ x = \frac{1}{3} \end{aligned}$$

When $x = \frac{1}{3}$, $y = \frac{1}{3}$ and when $x = 0$, $y = 1$. Thus we have four critical pionts: $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(\frac{1}{3}, \frac{1}{3})$. Now we test each; first we find second partials:

$$A = \frac{\partial^2 z}{\partial x^2} = 2y, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 2y + 2x - 1, \quad C = \frac{\partial^2 z}{\partial y^2} = 2x.$$

At $(0, 0)$, $A = 0,\ B = -1,\ C = 0$, so $B^2 - AC > 0$ and we have a saddle point;

At $(1, 0)$, $A = 0,\ B = 1,\ C = 2$, so $B^2 - AC > 0$ and we have a saddle point;

At $(0, 1)$, $A = 2,\ B = 1,\ C = 0$, so $B^2 - AC > 0$ and we have a saddle point;

At $(\frac{1}{3}, \frac{1}{3})$, $A = \frac{2}{3},\ B = \frac{1}{3},\ C = \frac{2}{3}$, so $B^2 - AC < 0$ and $A + C > 0$ and we have a relative minimum.

Note:

$$B^2 - AC = z_{xy}^2 - z_{xx}z_{yy} = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = |H|$$

where H is called the *Hessian* matrix of z.

Example 27.4. *Find critical points and test for extrema on* $z = 2x^3 - xy^2 + 5x^2 - y^2$.

We have

$$\begin{aligned} \frac{\partial z}{\partial y} &= -2xy + 2y = 0 \\ -2y(x-1) &= 0 \\ y &= 0, \; x = 1 \end{aligned}$$

Substitute each into $\frac{\partial z}{\partial x} = 6x^2 - y^2 + 10x = 0$ and solve. When $y = 0$, $6x^2 + 10x = 0$ implies that $x = 0$ or $x = -\frac{5}{3}$. When $x = 1$, $16 - y^2 = 0$ implies that $y = \pm 4$. Thus we have four critical pionts: $(0,0)$, $(1,4)$, $(1,-4)$, and $(-\frac{5}{3}, 0)$. Now we test each; first we find second partials:

$$A = \frac{\partial^2 z}{\partial x^2} = 12x + 10, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -2y, \quad C = \frac{\partial^2 z}{\partial y^2} = -2x + 2.$$

At $(0,0)$, $A = 10, \; B = 0, \; C = 2$, so $|H| < 0$ and $A + C > 0$ and we have a relative minimum;

At $(1,4)$, $A = 22, \; B = -8, \; C = 0$, so $|H| > 0$ and we have a saddle point;

At $(1,-4)$, $A = 22, \; B = -8, \; C = 0$, so $|H| > 0$ and we have a saddle point;

At $(-\frac{5}{3}, 0)$, $A = -10, \; B = 0, \; C = \frac{16}{3}$, so $|H| > 0$ and we have a saddle point.

Example 27.5. *Use the above theory to find maxima and minima of* $y = x^3 - 3x$.

HOMEWORK FOR DAY 27. Page 158, #4 parts b, c, d, and g; Page 159 #5 parts a, b, and e

HOMEWORK ANSWERS. #4 b) $\frac{\partial z}{\partial x} = 2x = 0$ implies $x = 0$. $\frac{\partial z}{\partial y} = 2y = 0$ implies $y = 0$. The only critical point is $(0, 0)$.

$$A = \frac{\partial^2 z}{\partial x^2} = 2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 z}{\partial y^2} = 2.$$

So $B^2 - AC < 0$ and $A + C > 0$, thus $(0, 0)$ is a relative minimum.

#4 c) $\frac{\partial z}{\partial x} = 4x - y - 3 = 0$ implies that $y = 4x - 3$; plugging this into $\frac{\partial z}{\partial y} = -x - 6y + 7 = 0$ gives $-25x + 25 = 0$ so that $x = 1$; hence $y = 1$, and $(1, 1)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = 4, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -1, \quad C = \frac{\partial^2 z}{\partial y^2} = -6.$$

So $B^2 - AC > 0$, thus $(1, 1)$ is a saddle point.

#4 d) $\frac{\partial z}{\partial x} = -3x = 0$ and $\frac{\partial z}{\partial y} = -7y = 0$ imply that $(0, 0)$ is the only critical point.

$$A = \frac{\partial^2 z}{\partial x^2} = -3, \quad B = \frac{\partial^2 z}{\partial x \partial y} = 0, \quad C = \frac{\partial^2 z}{\partial y^2} = -7.$$

So $B^2 - AC < 0$ and $A + C < 0$, thus $(0, 0)$ is a relative maximum.

#4 g) $\frac{\partial z}{\partial x} = 2x - 2\sin y - 2\cos y = 0$ implies $x = \sin y + \cos y$. We substitute into $\frac{\partial z}{\partial y}$:

$$\begin{aligned} \frac{\partial z}{\partial y} &= -2x(\cos y - \sin y) = 0 \\ -2(\sin y + \cos y)(\cos y - \sin y) &= 0 \\ \sin^2 y &= \cos^2 y \\ y &= \frac{\pi}{4} + k\pi,\ \frac{3\pi}{4} + k\pi,\ k \in \mathbb{Z} \end{aligned}$$

When $y = \frac{\pi}{4} + k\pi$, $x = \pm\sqrt{2}$ and when $y = \frac{3\pi}{4} + k\pi$, $x = 0$. Hence, we have infinitely many critical points of the form $\left(\pm\sqrt{2},\ \frac{\pi}{4} + k\pi\right)$, $\left(0,\ \frac{3\pi}{4} + k\pi\right)$.

$$A = \frac{\partial^2 z}{\partial x^2} = 2, \quad B = \frac{\partial^2 z}{\partial x \partial y} = -2\cos y + 2\sin y, \quad C = \frac{\partial^2 z}{\partial y^2} = 2x(\sin y + \cos y).$$

At $\left(\pm\sqrt{2},\ \frac{\pi}{4} + k\pi\right)$ we have $B^2 - AC < 0$ and $A + C > 0$ so these points are relative minima.

At $\left(0,\ \frac{3\pi}{4}+k\pi\right)$ we have $B^2-AC>0$ so these points are all saddle points.

#5 a) $\frac{\partial z}{\partial x}=-2xe^{-x^2-y^2}=0$ and $\frac{\partial z}{\partial y}=-2ye^{-x^2-y^2}=0$ imply $(0,0)$ is the only critical point.

$$\begin{aligned} A &= \frac{\partial^2 z}{\partial x^2} = -2e^{-x^2-y^2}+4x^2e^{-x^2-y^2} = -2, \\ B &= \frac{\partial^2 z}{\partial x \partial y} = 4xye^{-x^2-y^2} = 0, \\ C &= \frac{\partial^2 z}{\partial y^2} = -2e^{-x^2-y^2}+4y^2e^{-x^2-y^2} = -2. \end{aligned}$$

So $B^2-AC<0$ and $A+C<0$, thus $(0,0)$ is a relative maximum.

#5 b) $\frac{\partial z}{\partial x}=4x^3=0$ and $\frac{\partial z}{\partial y}=-4y^3=0$ imply $(0,0)$ is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=12x^2=0, \quad B=\frac{\partial^2 z}{\partial x\partial y}=0, \quad C=\frac{\partial^2 z}{\partial y^2}=-12y^2=0.$$

So $B^2-AC=0$ and this test fails! Only by graphing level curves do we find that the lines $y=x$ and $y=-x$ intersect at the origin, thus confirming that $(0,0)$ is a saddle point.

#5 e) $\frac{\partial z}{\partial x}=2x-y=0$ and $\frac{\partial z}{\partial y}=-x+2y=0$ imply that $(0,0)$ is the only critical point.

$$A=\frac{\partial^2 z}{\partial x^2}=2, \quad B=\frac{\partial^2 z}{\partial x\partial y}=-1, \quad C=\frac{\partial^2 z}{\partial y^2}=2.$$

So $B^2-AC<0$ and $A+C>0$, thus $(0,0)$ is a relative minimum.

28 Section 2.20, Absolute Extrema; Extrema with Constraints; The Lagrange Multiplier

Objective. *Students will determine absolute extrema of functions. Students will determine extrema of functions on a bounded set using a Lagrange multiplier.*

From single-variable calculus, we have

Theorem 28.1. *If $f(x)$ is continuous on the* closed *interval $a \le x \le b$, then $f(x)$ has both an absolute maximum and absolute minimum on $[a, b]$; in other words, $f(x)$ is* bounded *on the interval $[a, b]$.*

Of course, there is an analogue for two-variable:

Theorem 28.2. *Let D be a bounded domain of the xy-plane. Let $f(x, y)$ be defined and continuous in the closed region E formed of D plus its boundary. Then f has an absolute maximum and an absolute minimum in E.*

—[[Larson 104]]—

Example 28.1. *Find the absolute maximum and minimum of $z = x^2 + 2y^2 - x$ on the set $x^2 + y^2 \le 1$.*

Since $\frac{\partial z}{\partial x} = 2x - 1$, $\frac{\partial z}{\partial y} = 4y$, we have a critical point at $(\frac{1}{2}, 0)$; at this point, $z = -\frac{1}{4}$. Now, on the boundary, we substitute the boundary equation into z to get $z = 2 - x - x^2 = (2 + x)(1 - x)$ for $-1 \le x \le 1$. This function has the critical points $x = -1, -\frac{1}{2}, 1$. At these x-values, $z = 2, \frac{9}{4}, 0$. Hence the absolute maximum is $\frac{9}{4}$ at $(-\frac{1}{2}, 0)$, and the absolute minimum is $-\frac{1}{4}$ at $(\frac{1}{2}, 0)$. (Since the minimum occurs inside the set, it is also a relative minimum.)

Consider maximizing $w = f(x, y, z)$ where $g(x, y, z) = h(x, y, z) = 0$ are given as constraints. In other words, we find (x, y, z) along the curve defined by the intersection of g and h where f is at a maximum. At a maximum, the derivative of f along this curve must be zero, but this is the component of ∇f along the tangent; thus, ∇f is in the plane normal to the curve. Moreover, ∇g and ∇h are in the plane. Hence, $\nabla f, \nabla g, \nabla h$ are coplanar, so there are constants λ_1, λ_2 such that

$$\nabla f + \lambda_1 \nabla g + \lambda_2 \nabla h = \mathbf{0}.$$

The constant λ is called a *Lagrange multiplier.*

Example 28.2. *Find the extreme values of $z = x + 2y$ on the circle $x^2 + y^2 = 1$.*

Since $1 + 2\lambda x = 0$, $\lambda = -\frac{1}{2x}$; hence, $2 + 2\lambda y = 0$ becomes $2x = y$. Since $x^2 + y^2 = 1$, we have $x^2 + (2x)^2 = 1$ so that $x = \pm\sqrt{\frac{1}{5}}$; then $y = \pm 2\sqrt{\frac{1}{5}}$. The maximum is then $z = \sqrt{\frac{1}{5}} + 4\sqrt{\frac{1}{5}} = \sqrt{5}$ and the minimum is $z = -\sqrt{\frac{1}{5}} - 4\sqrt{\frac{1}{5}} = -\sqrt{5}$.

Example 28.3. *Find the point on the surface $x^2 + xy - z^2 + 4 = 0$ that is closest to the origin.*

Here, we minimize the distance function $\sqrt{x^2 + y^2 + z^2}$, which is the same as minimizing $x^2 + y^2 + z^2$. Then, since $2z - 2z\lambda = 0$ implies $\lambda = 1$, we use this value in the equation $2x + \lambda(2x + y) = 0$ to get that $y = -4x$. Using this relation and the value of λ in the equation $2y + \lambda x = 0$, we find that $x = y = 0$ and $z = \pm 2$. Hence, there are two points closest to the surface: $(0, 0, 2)$ and $(0, 0, -2)$.

Example 28.4. *Find the critical points of $w = xyz$ if $x^2 + y^2 + z^2 = 1$.*

We have

$$\begin{cases} yz + 2\lambda x = 0 \\ xz + 2\lambda y = 0 \\ xy + 2\lambda z = 0 \end{cases} \Rightarrow \begin{cases} xyz + 2\lambda x^2 = 0 \\ xyz + 2\lambda y^2 = 0 \\ xyz + 2\lambda z^2 = 0 \end{cases}$$

Adding equations, we have

$$\begin{aligned} 3xyz + 2\lambda(x^2 + y^2 + z^2) &= 0 \\ 3xyz + 2\lambda &= 0 \\ \lambda &= -\frac{3}{2}(xyz) \end{aligned}$$

Using this value of λ, we then have

$$\begin{cases} yz - 3x^2yz = 0 \\ xz - 3xy^2z = 0 \\ xy - 3xyz^2 = 0 \end{cases} \Rightarrow \begin{cases} yz(1 - 3x^2) = 0 \\ xz(1 - 3y^2) = 0 \\ xy(1 - 3z^2) = 0 \end{cases}$$

Hence, we have either two variables are 0 and one is ± 1, or all are $\pm\sqrt{\frac{1}{3}}$. Therefore, we have 14 critical points.

Example 28.5. *Find the extrema of $z = x + y$ given the constraint $x^2 + y^2 \leq 1$.*

We use the constraint to simplify z to the single variable x: $z = x + \sqrt{1 - x^2}$. Then we use single-variable methods to get that z has critical points $x = \pm\sqrt{\frac{1}{2}}$. Hence, the max is $\sqrt{2}$ and the min is $-\sqrt{2}$.

HOMEWORK FOR DAY 28. Page 159, #6 parts a, b, and c, #8 part b

Homework Answers. #6

a) We have $3+2\lambda x=0$, so $\lambda=-\frac{3}{2x}$. Thus, $4+2\lambda y=0$ becomes $4-\frac{3y}{x}=0$; so $y=\frac{4}{3}x$. Using this in the constraint gives $x^2+\left(\frac{4}{3}x\right)^2=1$, from which we see that $x=\pm\frac{3}{5}$ and $y=\pm\frac{4}{5}$. The maximum is $z=5$ at the point $(\frac{3}{5},\frac{4}{5})$ and the minimum is $z=-5$ at the point $(-\frac{3}{5},-\frac{4}{5})$.

b) We have $2x+4\lambda x^3=0$, so $\lambda=-\frac{1}{2x^2}$. Thus

$$\begin{aligned}
2y+4\lambda y^3&=0\\
2y-\frac{2y^3}{x^2}&=0\\
2y(x^2-y^2)&=0\\
y=0,\ x&=\pm y
\end{aligned}$$

According to the constraint equation, when $y=0$, $x=\pm1$; when $x=\pm y$, $x=y=\pm\sqrt[4]{\frac{1}{2}}$. The maximum is $\sqrt{2}$ at the point $\left(\sqrt[4]{\frac{1}{2}},\sqrt[4]{\frac{1}{2}}\right)$ and the minimum is $z=1$ at the points $(\pm1,0)$.

c) We have $2x+24+2\lambda x=0$, so $\lambda=-\frac{x+12}{x}$. Thus

$$\begin{aligned}
24+16y+2\lambda y&=0\\
24+16y-\frac{2y(x+12)}{x}&=0\\
12x+7xy-12y&=0\\
y(7x-12)&=-12x\\
y&=\frac{12x}{12-7x}
\end{aligned}$$

According to the constraint equation,

$$\begin{aligned}
x^2+\left(\frac{12x}{12-7x}\right)^2&=25\\
x^2+\frac{144x^2}{144-168x+49x^2}&=25\\
49x^4-168x^2-937x^2+4200x-3600&=0\\
(x-3)(x-4)(49x^2+175-300)&=0
\end{aligned}$$

Using the quadratic formula, we see the roots are

$$x = 3, 4, \frac{-25 \pm 5\sqrt{73}}{14}$$

But the constraint equation is in squares; so the negative of the roots also give critical points. Hence, the maximum is 425 at the points $(\pm 3, \pm 4)$, and the minimum is -200 at the points $(\pm 4, \mp 3)$.

#8 b) Since $\frac{\partial z}{\partial x} = y$ and $\frac{\partial z}{\partial y} = x$, we have the only critical point in the set is $(0, 0)$. On the boundary of $x^2 + y^2 \leq 1$, we have $z = x\sqrt{1 - x^2}$. Thus, $z' = \sqrt{1 - x^2} - x^2(1 - x^2)^{-1/2} = 0$, or $x = \pm\sqrt{\frac{1}{2}}$, which gives $y = \pm\sqrt{\frac{1}{2}}$. Thus, there is an absolute maximum of $\frac{1}{2}$ at $\left(\pm\sqrt{\frac{1}{2}}, \pm\sqrt{\frac{1}{2}}\right)$ and an absolute minimum of $-\frac{1}{2}$ at the point $\left(\pm\sqrt{\frac{1}{2}}, \mp\sqrt{\frac{1}{2}}\right)$.

29 Section 2.21, Maxima and Minima of Quadratic Forms on the Unit Sphere

Objective. *Students will determine maxima and minima of quadratic forms on the unit sphere. Students will determine whether a given quadratic form is positive definite.*

The most important application of Lagrange multipliers is in extrema of quadratic forms on the unit sphere. If we proceed with $f(x, y) = ax^2 + 2bxy + cy^2$ as before, with the constraint $g(x, y) = 1 - x^2 - y^2$, we have $f + \lambda g = ax^2 + 2bxy + cy^2 + \lambda(1 - x^2 - y^2)$, so that

$$2ax + 2by - 2\lambda x = 0$$
$$2bx + 2cy - 2\lambda y = 0$$

But these are equivalent to $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x}$ must meet the constraint; note that $||\mathbf{x}|| = 1$. Therefore the Lagrange multiplier is actually the eigenvalue of A, and $\mathbf{x}$ is an eigenvector of A.

Hence, the eigenvalues are the critical points of f on the unit sphere: particularly, the abs max of f on the unit sphere is the largest eigenvalue; the abs min of f is the smallest eigenvalue. Also, the corresponding unit eigenvectors are the values that give the critical points.

Example 29.1. *Find the extrema of $x^2 + 4xy - 2y^2$ on the unit circle.*

The matrix is $\left[\begin{smallmatrix} 1 & 2 \\ 2 & -2 \end{smallmatrix}\right]$ with eigenvalues -3 and 2 and unit eigenvectors $\frac{1}{\sqrt{5}}\langle 1, -2\rangle$ and $\frac{1}{\sqrt{5}}\langle 2, 1\rangle$. Hence, the max of f is 2 given by $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, and the min is -3 given by $(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.

Moreover, f has a positive minimum on the unit sphere iff f is positive for all $\mathbf{x}$. Such a quadratic form is positive definite. Also, f has a negative maximum on the unit sphere iff f is negative for all $\mathbf{x}$. Such a quadratic form is negative definite.

Hence, we have

Theorem 29.1. *A quadratic form is positive definite iff all its eigenvalues are positive.*

HOMEWORK FOR DAY 29. Page 159, #9

30 Sections 3.2 and 3.3, Vector Fields, Scalar Fields, and Gradient Fields

Objective. *Students will define and investigate the properties of vector fields, scalar fields, and gradient fields.*

A domain D in space is called a *vector field* if each point of D can be assigned a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

By convention, vectors in a vector field are drawn with the tail, not the head, at the point of evaluation. —[[Stewart 54]]—

Example 30.1. *Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $\mathbf{v}$ is a vector pointing away from the origin.*

Example 30.2. *Let $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$. Then $\mathbf{v}$ characterizes a counterclockwise rotation about the z-axis.*

Example 30.3. *Let*

$$\mathbf{v} = \frac{1}{\left[(x+1)^2 + y^2\right]\left[(x-1)^2 + y^2\right]}\left[2(x^2 - y^2 - 1)\mathbf{i} + 4xy\mathbf{j}\right].$$

Then $\mathbf{v}$ is the electric force field from two oppositely charged wires, one at $(-1, 0)$ and the other at $(1, 0)$.

—[[Stewart 55]]— A domain D in space is called a *scalar field* if each point in D can be assigned a scalar (i.e., temperature at each point in a room).

Example 30.4. *Let $f(x, y, z) = e^{x+y-z}$. Then there is a scalar at each point in space.*

Note that any vector field can give a scalar field (i.e., $||\mathbf{v}||$), and any scalar field can give a vector field (i.e., ∇f). —[[Stewart 56, 57]]—

A scalar field that has a defined and differentiable function $f(x, y, z)$ on D so that ∇f exists at each point is called a *gradient field*.

The gradient symbol ∇ is not a function; it is a *vector differential operator*. In other words, ∇ is defined to be the vector operation $\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ applied to a function. This is a unary operation. Note that ∇ has the following properties:

$$\nabla(f + g) = \nabla f + \nabla g, \qquad \nabla(fg) = f\nabla g + g\nabla f,$$

and if c is a constant,

$$\nabla(cf) = c\nabla f.$$

Since the gradient is an operator, we have

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) f$$

where the "multiplication" is actually differentiation.

Example 30.5. *Sketch the gradient field of* $f(x, y, z) = e^{x+y-z}$.

$\nabla f = e^{x+y-z}(\mathbf{i} + \mathbf{j} - \mathbf{k})$, so the gradient field are all vectors in the same direction of magnitude $\sqrt{3}(e^{x+y-z})$.

HOMEWORK FOR DAY 30. Page 180, #1 part c, #2 parts a and b, #3 parts a and b, and #5

HOMEWORK ANSWERS. #1 c) The field resembles a counter-clockwise rotation about a cylinder.

#2 a) The level curves are hyperbolas.

#2 b) The level surphaces are sferes!

#3 a) $\nabla f = y\mathbf{i} + x\mathbf{j}$

#3 b) $\nabla f = 2x\mathbf{i} + 2y\mathbf{j} - 2z\mathbf{k}$

#5 Rewrite f as $f = \frac{1}{2}\log[(x-1)^2 + y^2] - \frac{1}{2}\log[(x+1)^2 + y^2]$. Then differentiate and find common denominators, and we have

$$\begin{aligned}\nabla f &= \left(\frac{x-1}{[(x-1)^2+y^2]} - \frac{x+1}{[(x-1)^2+y^2]}\right)\mathbf{i} + \left(\frac{y}{[(x-1)^2+y^2]} - \frac{y}{[(x-1)^2+y^2]}\right)\mathbf{j} \\ &= \frac{2x^2-2y^2-2}{[(x+1)^2+y^2]\,[(x-1)^2+y^2]}\mathbf{i} + \frac{4xy}{[(x+1)^2+y^2]\,[(x-1)^2+y^2]}\mathbf{j}\end{aligned}$$

31 Sections 3.4 and 3.5, The Divergence and The Curl

Objective. *Students will compute the divergence and the curl, and prove identities involving both.*

Given a vector field $\mathbf{F} \in D$, each element of $\mathbf{F}$ is

$$\mathbf{v} = v_1(x, y, z)\mathbf{i} + v_2(x, y, z)\mathbf{j} + v_3(x, y, z)\mathbf{k}.$$

If the partials are defined in D, we may arrange the matrix

$$\begin{bmatrix} \dfrac{\partial v_1}{\partial x} & \dfrac{\partial v_1}{\partial y} & \dfrac{\partial v_1}{\partial z} \\ \dfrac{\partial v_2}{\partial x} & \dfrac{\partial v_2}{\partial y} & \dfrac{\partial v_2}{\partial z} \\ \dfrac{\partial v_3}{\partial x} & \dfrac{\partial v_3}{\partial y} & \dfrac{\partial v_3}{\partial z} \end{bmatrix}$$

The sum of the diagonal elements (the trace) is the called *the divergence* of $\mathbf{v}$, denoted $\operatorname{div}\mathbf{v}$.

Example 31.1. *Find* $\operatorname{div}\mathbf{v}$ *if* $\mathbf{v} = x^3\mathbf{i} + 2xy\mathbf{j} - xyz^2\mathbf{k}$.

We have $\operatorname{div}\mathbf{v} = 3x^2 + 2x - 2xyz$.

Uses: $\operatorname{div}\mathbf{v}$ is the measure of the rate of decrease of density at a point in fluid dynamics. Also, the divergence of an electric force vector is $4\pi\rho$ where ρ is the charge density in the electric field.

Note that div is also an operator and in terms of operators, $\operatorname{div}\mathbf{v} = \nabla \cdot \mathbf{v}$.

Theorem 31.1. *The divergence operator has the following properties.*

$$\operatorname{div}(\mathbf{u} + \mathbf{v}) = \operatorname{div}\mathbf{u} + \operatorname{div}\mathbf{v} \tag{31.1}$$

$$\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \nabla f \cdot \mathbf{v} \tag{31.2}$$

Proof. First, we prove Eq. 31.1.

$$\begin{aligned} \operatorname{div}(\mathbf{u} + \mathbf{v}) &= \frac{\partial(u_1 + v_1)}{\partial x} + \frac{\partial(u_2 + v_2)}{\partial y} + \frac{\partial(u_3 + v_3)}{\partial z} \\ &= \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial y} + \frac{\partial u_3}{\partial z} + \frac{\partial v_3}{\partial z} \\ &= \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} + \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \\ &= \operatorname{div}\mathbf{u} + \operatorname{div}\mathbf{v} \end{aligned}$$

Next, we prove Eq. 31.2 with the aid of the product rule.

$$
\begin{aligned}
\operatorname{div}(f\mathbf{v}) &= \frac{\partial(fv_1)}{\partial x} + \frac{\partial(fv_2)}{\partial y} + \frac{\partial(fv_3)}{\partial z} \\
&= \frac{\partial f}{\partial x}v_1 + f\frac{\partial v_1}{\partial x} + \frac{\partial f}{\partial y}v_2 + f\frac{\partial v_2}{\partial y} + \frac{\partial f}{\partial z}v_3 + f\frac{\partial v_3}{\partial z} \\
&= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \frac{\partial f}{\partial x}v_1 + \frac{\partial f}{\partial y}v_2 + \frac{\partial f}{\partial z}v_3 \\
&= f\operatorname{div}\mathbf{v} + \nabla f\cdot\mathbf{v}
\end{aligned}
$$

□

For the other six elements in the matrix above, we define *the curl* of $\mathbf{v}$ by

$$
\begin{aligned}
\operatorname{curl}\mathbf{v} &= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}
\end{aligned}
$$

The curl measures angular motion of a fluid.

Example 31.2. *Find* $\operatorname{curl}\mathbf{v}$ *if* $\mathbf{v} = x^3\mathbf{i} + 2xy\mathbf{j} - xyz^2\mathbf{k}$.

We have $\operatorname{curl}\mathbf{v} = -xz^2\mathbf{i} + yz^2\mathbf{j} + 2y\mathbf{k}$.

Note that the curl is an operator and in terms of operators, we have $\operatorname{curl}\mathbf{v} = \nabla\times\mathbf{v}$.

Theorem 31.2. *The curl operator has the following properties.*

$$\operatorname{curl}(\mathbf{u}+\mathbf{v}) = \operatorname{curl}\mathbf{u} + \operatorname{curl}\mathbf{v} \tag{31.3}$$

$$\operatorname{curl}(f\mathbf{v}) = f\operatorname{curl}\mathbf{v} + \nabla f\times\mathbf{v} \tag{31.4}$$

Proof. We prove Eq. 31.3 first.

$$\begin{aligned}
\operatorname{curl}(\mathbf{u}+\mathbf{v}) &= \left(\frac{\partial(u_3+v_3)}{\partial y} - \frac{\partial(u_2+v_2)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(u_1+v_1)}{\partial z} - \frac{\partial(u_3+v_3)}{\partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial(u_2+v_2)}{\partial x} - \frac{\partial(u_1+v_1)}{\partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial u_3}{\partial y} + \frac{\partial v_3}{\partial y} - \frac{\partial u_2}{\partial z} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u_1}{\partial z} + \frac{\partial v_1}{\partial z} - \frac{\partial u_3}{\partial x} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial x} - \frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial y}\right)\mathbf{k} \\
&= \operatorname{curl}\mathbf{u} + \operatorname{curl}\mathbf{v}
\end{aligned}$$

Next, we prove Eq. 31.4 by using the product rule.

$$\begin{aligned}
\operatorname{curl}(f\mathbf{v}) &= \left(\frac{\partial(fv_3)}{\partial y} - \frac{\partial(fv_2)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(fv_1)}{\partial z} - \frac{\partial(fv_3)}{\partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial(fv_2)}{\partial x} - \frac{\partial(fv_1)}{\partial y}\right)\mathbf{k} \\
&= \left(\frac{\partial f}{\partial y}v_3 + f\frac{\partial v_3}{\partial y} - \frac{\partial f}{\partial z}v_2 - f\frac{\partial v_2}{\partial z}\right)\mathbf{i} \\
&\quad + \left(\frac{\partial f}{\partial z}v_1 + f\frac{\partial v_1}{\partial z} - \frac{\partial f}{\partial x}v_3 - f\frac{\partial v_3}{\partial x}\right)\mathbf{j} \\
&\quad + \left(\frac{\partial f}{\partial x}v_2 + f\frac{\partial v_2}{\partial x} - \frac{\partial f}{\partial y}v_1 - f\frac{\partial v_1}{\partial y}\right)\mathbf{k} \\
&= f\left[\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}\right] \\
&\quad + \left(\frac{\partial f}{\partial y}v_3 - \frac{\partial f}{\partial z}v_2\right)\mathbf{i} + \left(\frac{\partial f}{\partial z}v_1 - \frac{\partial f}{\partial x}v_3\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}v_2 - \frac{\partial f}{\partial y}v_1\right)\mathbf{k} \\
&= f\operatorname{curl}\mathbf{v} + \nabla f \times \mathbf{v}
\end{aligned}$$

□

HOMEWORK FOR DAY 31. Page 185, #5 part a; Page 186, #12 parts a and b

HOMEWORK ANSWERS. #5 $\operatorname{curl}\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyx & x^2z & x^2y \end{vmatrix} = (x^2 - x^2)\mathbf{i} + (2xy - 2xy)\mathbf{j} + (2xz - 2xz)\mathbf{k} = \mathbf{0}$. Functions that satisfy $\nabla f = \mathbf{v}$ are of the form $f = x^2yz + c$ where c is a constant.

#12 a) Since $\mathbf{u}$ is a unit vector, we may write the components of $\mathbf{u}$ in terms of the direction angles:

$$\begin{aligned}(\mathbf{u}\cdot\nabla)f &= u_1\frac{\partial f}{\partial x} + u_2\frac{\partial f}{\partial y} + u_3\frac{\partial f}{\partial z} \\ &= \cos\alpha\frac{\partial f}{\partial x} + \cos\beta\frac{\partial f}{\partial x} + \cos\gamma\frac{\partial f}{\partial x} \\ &= \nabla_{\mathbf{u}}f\end{aligned}$$

#12 b) $[(\mathbf{i} - \mathbf{j})\cdot\nabla]f = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$.

32 Section 3.6, Combined Operations

Objective. *Students will prove identities that combine the divergence, the curl, and the gradient.*

The previous theorems imply that the gradient, the divergence, and the curl are *linear operators*; i.e., they satisfy the basic property of linear transformations: $L(c_1\mathbf{u} + c_2\mathbf{v}) = c_1L(\mathbf{u}) + c_2L(\mathbf{v})$.

We may combine these operations to create the following identities.

I) $\operatorname{curl} \nabla f = \mathbf{0}$.

Proof. We have

$$\operatorname{curl} \nabla f = \nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

in which each component consists of the difference of identical mixed partials; hence, $\operatorname{curl} \nabla f = \mathbf{0}$. □

Note that if $\operatorname{curl} \mathbf{v} = \mathbf{0}$, then there is a function f such that $\mathbf{v} = \nabla f$ (as in homework problem #5 from last time).

II) $\operatorname{div} \operatorname{curl} \mathbf{v} = 0$. (*The proof is left for homework.*) Note that if $\operatorname{div} \mathbf{v} = 0$, then there is a vector $\mathbf{u}$ such that $\mathbf{v} = \operatorname{curl} \mathbf{u}$.

III) $\operatorname{div}(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \operatorname{curl} \mathbf{u} - \mathbf{u} \cdot \operatorname{curl} \mathbf{v}$.

Proof. We have

$$\begin{aligned}
\operatorname{div}(\mathbf{u}\times\mathbf{v}) &= \operatorname{div}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\
&= \operatorname{div}\left[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\right] \\
&= \frac{\partial(u_2v_3 - u_3v_2)}{\partial x} - \frac{\partial(u_1v_3 - u_3v_1)}{\partial y} + \frac{\partial(u_1v_2 - u_2v_1)}{\partial z} \\
&= \frac{\partial u_2}{\partial x}v_3 + u_2\frac{\partial v_3}{\partial x} - \frac{\partial u_3}{\partial x}v_2 - u_3\frac{\partial v_2}{\partial x} - \frac{\partial u_1}{\partial y}v_3 - u_1\frac{\partial v_3}{\partial y} \\
&\quad + \frac{\partial u_3}{\partial y}v_1 + u_3\frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial z}v_2 + u_1\frac{\partial v_2}{\partial z} - \frac{\partial u_2}{\partial z}v_1 - u_2\frac{\partial v_1}{\partial z} \\
&= \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)v_1 + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial x}\right)v_2 + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)v_3 \\
&\quad - u_1\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - u_2\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_2}{\partial x}\right) - u_3\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \\
&= \mathbf{v}\cdot\operatorname{curl}\mathbf{u} - \mathbf{u}\cdot\operatorname{curl}\mathbf{v}
\end{aligned}$$

□

IV) $\operatorname{div}\nabla f = \nabla^2 f$.

Proof.

$$\begin{aligned}
\operatorname{div}\nabla f &= \operatorname{div}\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \nabla^2 f
\end{aligned}$$

□

V) $\operatorname{curl}\operatorname{curl}\mathbf{u} = \nabla(\operatorname{div}\mathbf{u}) - \nabla^2\mathbf{u}$.

HOMEWORK FOR DAY 32. Page 185, #6; Page 186, #11 parts b, c, and d

HOMEWORK ANSWERS. #6 This is easy.

#11 b) $\operatorname{div}(\nabla f \times f\nabla g) = f\nabla g \operatorname{curl} \nabla f - \nabla f \operatorname{curl}(f\nabla g) = (f\nabla g)(0) - (\nabla f)(0) = 0$.

#11 c) Since $\operatorname{curl} \nabla f = 0$, we have $\operatorname{curl}(\operatorname{curl} \mathbf{v} + \nabla f) = \operatorname{curl} \operatorname{curl} \mathbf{v} + \operatorname{curl} \nabla f = \operatorname{curl} \operatorname{curl} \mathbf{v}$.

#11 d) Since $\operatorname{div} \operatorname{curl} \mathbf{v} = 0$, we obtain $\operatorname{div}(\operatorname{curl} \mathbf{v} + \nabla f) = \operatorname{div} \operatorname{curl} \mathbf{v} + \operatorname{div} \nabla f = \nabla^2 f$.

33 Section 4.1, The Definite Riemann Integral

Objective. *Students will review single-variable integration.*

Definition: $\int_a^b f(x)\ dx = \lim_{h\to 0}\sum_{i=1}^{n} f(x_i^*)\Delta_i x$ where $a = x_0 < x_1 < \cdots < x_n = b$; $\Delta_i x = x_i - x_{i-1}$; $x_{i-1} \le x_i^* \le x_i$; and, $h = \max \Delta_i x$.

This Riemann integral exsits provided f is bounded: $|f(x)| \le M$ for some real M.

Antiderivatives (also called *primitives*) by substitution and parts:

Example 33.1. *Find* $\int \frac{dx}{1+\sqrt{x-1}}$ *where* $x \ge 1$.

We use substitution twice. Let $u = \sqrt{x-1}$; then $dx = 2u\ du$. Hence,

$$\int \frac{dx}{1+\sqrt{x-1}} = \int \frac{2u\ du}{1+u}$$

Now, let $w = 1+u$, so that $2u = 2w-2$ and $du = dw$.

$$\begin{aligned} &= \int \frac{2w-2}{w}\ dw = \int \left(2 - \frac{2}{w}\right) dw \\ &= 2w - \log w + C = 2\left(1+u-\log(1+u)\right) + C \\ &= 2\left(1+\sqrt{x-1} - \log\left(1+\sqrt{x-1}\right)\right) + C \end{aligned}$$

Example 33.2. *Evaluate* $\int_0^1 \arctan x\ dx$.

Consider the indefinite integral so we can find an antiderivative by parts. Let $u = \arctan x,\ dv = dx$ so that $du = \frac{1}{x^2+1},\ v = x$. Thus,

$$\begin{aligned} \int \arctan x\ dx &= x \arctan x - \int \frac{x}{x^2+1} \\ &= x\arctan x - \tfrac{1}{2}\log(x^2+1) + C \end{aligned}$$

Hence, the definite integral is

$$\begin{aligned} \int_0^1 \arctan x\ dx &= x\arctan x - \tfrac{1}{2}\log(x^2+1)\Big|_0^1 \\ &= \arctan 1 - \tfrac{1}{2}\log 2 - \arctan 0 + \tfrac{1}{2}\log 1 \\ &= \frac{\pi}{4} - \frac{\log 2}{2} \end{aligned}$$

Theorem 33.1 (The Mean Value Theorem for Integrals). *If f is continuous on the interval $a \le x \le b$, then for some $x^* \in [a,b]$ we have $\int_a^b f(x)\,dx = f(x^*)(b-a)$. In other words, the average (mean) value of f on the interval $a \le x \le b$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.*

Theorem 33.2 (The Fundamental Theorem of Calculus). *If f is continuous on the interval $a \le x \le b$, with $c \in [a,b]$, and F is an antiderivative of f, then*

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x) \tag{33.1}$$

and

$$\int_c^x f(t)\,dt = F(x) - F(c) \tag{33.2}$$

Proof. Proof of Eq. 33.1. [Apostol] Let h be a positive real number and let $x \in [a,b]$. Let the integral in (33.1) be denoted by $A(x)$. Then

$$\int_x^{x+h} f(t)\,dt = \int_c^{x+h} f(t)\,dt - \int_c^x f(t)\,dt = A(x+h) - A(x).$$

By the Mean Value Theorem for Integrals, we have $A(x+h) - A(x) = hf(x^*)$ for some $x^* \in [x, x+h]$. Hence,

$$\frac{A(x+h) - A(x)}{h} = f(x^*),$$

and, since $x \le x^* \le x+h$, we find that $f(x^*) \to f(x)$ as $h \to 0$. Thus, $A'(x)$ exists and is equal to $f(x)$.

Proof of Eq. 33.2. Let $A(x)$ denote the integral in 33.2. Since f is continuous, Eq. 33.1 implies that $A'(x) = f(x)$ over $[a,b]$; in other words, A is an antiderivative of f. Since two antiderivatives can only differ by a constant, we have $A(x) - F(x) = k$. When $x = c$, $A(c) = 0$ so that $-F(c) = k$. Therefore, $A(x) - F(x) = -F(c)$, or $A(x) = F(x) - F(c)$. □

Improper integrals are technically not Riemann integrals, but we can use limits to evaluate convergent improper integrals; clearly, there are divergent ones as well.

Example 33.3. *Evaluate* $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{1+x^2}}$.

We find the antiderivative by trigonometric substitution. Let $x = \tan\theta$, so that $dx = \sec^2\theta$. Then

$$\int \frac{dx}{x\sqrt{1+x^2}} = \int \frac{\sec^2\theta\ d\theta}{\tan\theta\sec\theta} = \int \frac{\sec\theta\ d\theta}{\tan\theta} = \int \csc\theta\ d\theta = \log\left(\csc\theta - \cot\theta\right)$$

Since $x = \tan\theta$, then $\csc\theta = \sqrt{x^2+1}/x$, $\cot\theta = 1/x$. Thus,

$$\int \frac{dx}{x\sqrt{1+x^2}} = \log\left(\frac{\sqrt{x^2+1}-1}{x}\right)$$

Now, we deal with the infinite limit:

$$\lim_{b\to\infty} \log\left(\frac{\sqrt{x^2+1}-1}{x}\right)\Bigg|_1^b = \lim_{b\to\infty}\left[\log\left(\frac{\sqrt{b^2+1}-1}{b}\right) - \log(\sqrt{2}-1)\right]$$
$$= \log 1 - \log(\sqrt{2}-1) = \log(\sqrt{2}+1)$$

If the answer had been infinite, this would have been a divergent integral.

We also have many numerical techniques – Simpson's Rule, Riemann Sums – but sufficiently accurate of the basic methods is the following.

Theorem 33.3 (The Trapezoid Rule). *If f is continuous over the interval $[a,b]$, then $\int_a^b f(x)\ dx$ can be approximated by*

$$\Delta_1 x\frac{f(a)+f(x_1)}{2} + \cdots + \Delta_n x\frac{f(x_{n-1})-f(b)}{2},$$

where $a = x_0 < x_1 < \cdots < x_n = b$ and $\Delta_i x = x_i - x_{i-1}$. If the subinterval widths $\Delta_i x$ are all equal, then the approximation becomes

$$\frac{b-a}{2n}\left[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)\right].$$

Moreover, if $|f''(x)| \le L$ on the interval, then the approximation differs from the integral by at most $\dfrac{L(b-a)^3}{12n^2}$.

HOMEWORK FOR DAY 33. Page 219, #1 parts b, c, and d, #2 parts a and d, #3 parts b and c; Page 220, #6 part a, and the following: Find the error bound in using the trapezoid rule to approximate $\displaystyle\int_{-1}^2 (x^3 - 2x^2 + 3)\ dx$ using 6 subintervals.

HOMEWORK ANSWERS. #1 b) $\displaystyle\int \frac{x\ dx}{1+x^4} = \frac{1}{2}\int \frac{2x\ dx}{1+(x^2)^2} = \frac{1}{2}\arctan x^2 + C$

#1 c) We use partial fractions.

$$\int \frac{dx}{(x-1)(x-2)} = \int \left(\frac{-1}{x-1} + \frac{1}{x-2}\right) dx$$
$$= -\log(x-1) + \log(x-2) + C = \log\left(\frac{2-x}{x-1}\right) + C$$

#2 a) The integral represents a quarter-circle; it is equal to $\pi/4$.

#3 b) $\displaystyle\lim_{b\to\infty}\int_0^b e^{-x}\ dx = \lim_{b\to\infty}(-e^{-x})\Big|_0^b = \lim_{b\to\infty}(-e^{-b}+1) = 1.$

#3 c) We use parts to find the antiderivative, and apply L'Hopital's Rule to evaluate the limit.

$$\int_0^1 \log x\ dx = \lim_{b\to 0^+}\int_b^1 \log x\ dx = \lim_{b\to 0^+}(x\log x - x)\Big|_b^1$$
$$= \lim_{b\to 0^+}(-1 - b\log b + b) = \lim_{b\to 0^+}\left(-1 - \frac{\log b}{1/b} + b\right)$$
$$= \lim_{b\to 0^+}\left(-1 - \frac{1/b}{-1/b^2} + b\right)$$
$$= \lim_{b\to 0^+}(-1 + 2b) = -1$$

Trapezoid problem: Since $f''(x) = 6x - 4 \le 8$ on $-1 \le x \le 2$, we have the error as

$$\frac{8(2+1)^3}{12(6^2)} = \frac{1}{2}.$$

34 Section 4.2, Numerical Evaluation of Indefinite Integrals; Elliptic Integrals

Objective. *Students will tabulate numerical values of indefinite integrals in order to approximate the antiderivatives. Students will investigate elliptic integrals.*

We know $\int e^{x^2}\,dx$ has no "closed-form" antiderivative. But thanks to the Fundamental Theorem, we can easily approximate the integral by calculating values of $F(x) = \int_0^x e^{t^2}\,dt$ for $0 \le x \le 1$ with $\Delta x = 0.1$. This is done for $F(x) = \int_0^x e^{\sin t}\,dt$ on page 222.

We may graph the points of F and use the regression capabilities of the graphing calculator to gain an approximate antiderivative of $e^{\sin t}$. An approximate cubic function for F on $[0,1]$ is $F(x) \approx 0.054x^3 + 0.595x^2 + 0.983x$. Note that $F'(0.2) \approx 1.23$; compare to $e^{\sin 0.2} \approx 1.22$.

The most useful of these types of integrals – whose antiderivatives are approximate – are the *elliptic integrals*, where $0 < k^2 < 1,\ a \neq 0,\ a^2 \neq k^2$:

- 1st kind – $y = \displaystyle\int_0^x \frac{dt}{\sqrt{1 - k^2\sin^2 t}}$
- 2nd kind – $y = \displaystyle\int_0^x \sqrt{1 - k^2\sin^2 t}\,dt$
- 3rd kind – $y = \displaystyle\int_0^x \frac{dt}{\sqrt{1 - k^2\sin^2 t}(1 + a^2\sin^2 t)}$

The second kind arises from the arc length of an ellipse, hence the name.

If $f(x)$ is a rational function of the form $x/P(x)$ where $P(x)$ is a polynomial of degree 3 or 4, then $\int x/P(x)\,dx$ can be expressed as an elementary function plus an elliptic integral. So indefinite integrals of all kinds are numerically useful and can even be used to define functions. For example, let $y = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$. Then $y = \arcsin x$, or $x = \sin y$. Although more complicated, all sine properties can be derived from this definition.

HOMEWORK FOR DAY 34. Page 223, #1 part b–and find a cubic approximation for the antiderivative; Page 224, #3 parts a and b, #4, (#8 is extra credit)

HOMEWORK ANSWERS. #1 b)

x	$\int_0^x e^{-t^2}\ dt$
0	0
0.1	0.1
0.2	0.197
0.3	0.291
0.4	0.380
0.5	0.461
0.6	0.535
0.7	0.601
0.8	0.658
0.9	0.706
1	0.747

Cubic approximation is $-0.13x^3 - 0.156x^2 + 1.033x$ for $0 \leq x \leq 1$.

#3 a) This is easy.

#3 b) The integrand is continuous, so it's area is continuous; the derivative follows from the Fundamental Theorem.

#4 We have $\frac{dx}{d\phi} = -a\sin\phi$, $\frac{dy}{d\phi} = b\cos\phi$. Hence,

$$\begin{aligned} L &= \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\ d\phi = \int_0^\alpha \sqrt{a^2\sin^2\phi + b^2 - b^2\sin^2\phi}\ d\phi \\ &= \int_0^\alpha \sqrt{(a^2-b^2)\sin^2\phi + b^2}\ d\phi = b\int_0^\alpha \frac{1}{b}\sqrt{b^2-(b^2-a^2)\sin^2\phi}\ d\phi \\ &= b\int_0^\alpha \sqrt{1 - \frac{b^2-a^2}{b^2}\sin^2\phi\ d\phi} \end{aligned}$$

#8 a) This is also easy.

#8 b) $\operatorname{erf}(-x) = \int_0^{-x} e^{-t^2}\ dt = -\int_{-x}^0 e^{-t^2}\ dt$; but the integrand is even, so $\int_{-x}^0 e^{-t^2}\ dt = \int_0^x e^{-t^2}\ dt$. Hence,

$$\operatorname{erf}(-x) = -\int_{-x}^0 e^{-t^2}\ dt = -\int_0^x e^{t^2}\ dt = -\operatorname{erf}(x).$$

#8 c) $\int_0^\infty e^{-t}\ dt$ converges to 1 and since $e^{-t^2} < e^{-t}$, we have $\int_0^\infty e^{-t^2}\ dt < \int_0^\infty e^{-t}\ dt$. Noting that e^{-t^2} is even gives $-1 < \operatorname{erf}(x) < 1$.

35 Section 4.3 A, Double Integrals

Objective. *Students will use Fubini's Theorem to evaluate double integrals by switching the order of integration.*

The antiderivative of a two-variable function $f(x, y)$ is a *double integral* $\iint_R f(x, y)\ dx\ dy$ over a region R in the xy-plane. The definition is

$$\iint\limits_R f(x,y)\ dx\ dy = \lim_{h \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta_i A$$

where $\Delta_i A$ is the area of the ith rectangle in R, h is the maximum diagonal of the n rectangles, and (x_i^*, y_i^*) is an arbitrary point in the ith rectangle. —[[THOMAS 12.6]]—

The basic interpretation of a double integral is the volume beneath the surface f over the region R (similar to the area beneath the curve f over the interval I). —[[STEWART 48, LARSON 107, STEWART 49]]—

We may interpret the region R as $a \leq x \leq b$, in which case $g_1(x) \leq y \leq g_2(x)$; or as $a \leq y \leq b$, in which case $h_1(y) \leq x \leq h_2(y)$. We assume that g_1, g_2, h_1, h_2 are all continuous. —[[LARSON 106]]—

Evaluation of a double integral is done by reducing it to an *iterated integral*: $\int_a^b \left[\int_{g_1}^{g_2} f(x,y)\ dy\right] dx$. The inner integral represents the area of a cross section of the solid perpendicular to the x-axis. Hence, $\int_a^b [\text{Area}]\ dx = \text{Volume}$. —[[THOMAS 12.8, 12.4, 12.5, LARSON 108]]—

Theorem 35.1 (Fubini's Theorem). *If $f(x, y)$ is continuous in a closed region R described by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, then*

$$\iint\limits_R f(x,y)\ dx\ dy = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx$$

where the inner integral is a continuous function of x. If R is described by $a \leq y \leq b$ and $h_1(y) \leq x \leq h_2(y)$, then

$$\iint\limits_R f(x,y)\ dx\ dy = \int_a^b \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$$

where the inner integral is a continuous function of y.

—[[STEWART 49 AGAIN]]—

Example 35.1. *Find* $\iint_R (16 - x^2 - 2y^2)\ dx\ dy$ *if* R *is the region defined by* $0 \le x, y \le 2$.

$$\begin{aligned}\iint\limits_R (16 - x^2 - 2y^2)\ dx\ dy &= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\ dx\ dy \\ &= \int_0^2 16x - \tfrac{1}{3}x^3 - 2xy^2\Big|_0^2\ dy \\ &= \int_0^2 (\tfrac{88}{3} - 4y^2)\ dy = \tfrac{88}{3}y - \tfrac{4}{3}y^3\Big|_0^2 \\ &= \tfrac{176}{3} - \tfrac{32}{3} = 48\end{aligned}$$

—[[THOMAS 12.1, 12.2]]—

Example 35.2. *Find* $\iint_R 2xy\ dx\ dy$ *where* R *is the region bounded by* $x^2 + y^2 = 1$ *and* $x + y = 1$.

We do this two ways: with respect to y then x, and vice versa.

$$\begin{aligned}\iint\limits_R 2xy\ dy\ dx &= \int_0^1 \int_{1-x}^{\sqrt{1-x^2}} 2xy\ dy\ dx = \int_0^1 xy^2\Big|_{1-x}^{\sqrt{1-x^2}}\ dx \\ &= \int_0^1 [x(1-x^2) - x(1-x)]\ dx = \int_0^1 (x^2 - x^3)\ dx \\ &= \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\Big|_0^1 = \tfrac{1}{12}\end{aligned}$$

Equivalently,

$$\begin{aligned}\iint\limits_R 2xy\ dx\ dy &= \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} 2xy\ dx\ dy = \int_0^1 x^2y\Big|_{1-y}^{\sqrt{1-y^2}}\ dy \\ &= \int_0^1 [y(1-y^2) - y(1-y)]\ dy = \int_0^1 (y^2 - y^3)\ dy \\ &= \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\Big|_0^1 = \tfrac{1}{12}\end{aligned}$$

Example 35.3. *Find* $\iint_R f(x, y)\ dx\ dy$ *if* R *is the region defined by* $0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2}$ *and* $f(x, y) = xy$.

$$\iint\limits_R xy\ dx\ dy = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\ dy\ dx = \int_0^1 \tfrac{1}{2}xy^2\Big|_0^{\sqrt{1-x^2}}\ dx$$
$$= \int_0^1 \tfrac{1}{2}x(1-x^2)\ dx = \tfrac{1}{2}\int_0^1 (x-x^3)\ dx$$
$$= \tfrac{1}{2}\left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)\Big|_0^1 = \tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{4}\right) = \tfrac{1}{8}$$

Example 35.4. *Find* $\iint_R (x+2y)\ dx\ dy$ *where* R *is the triangle with vertices* $(2,1)$, $(4,1)$, *and* $(2,2)$.

We sketch the triangular region and find that $2 \le x \le 4,\ 1 \le y \le 3 - \frac{1}{2}x$. Thus,

$$\iint_R (x+2y)\ dx\ dy = \int_2^4 \int_1^{3-\frac{1}{2}x} (x+2y)\ dy\ dx$$
$$= \int_2^4 (xy+y^2)\Big|_1^{3-\frac{1}{2}x} dx$$
$$= \int_2^4 \left(3x - \tfrac{1}{2}x^2 - 9 - 3x + \tfrac{1}{4}x^4 - x - 1\right)\ dx$$
$$= -\tfrac{1}{12}x^3 - \tfrac{1}{2}x^2 + 8x\Big|_2^4 = \tfrac{16}{3}$$

Example 35.5. *Find the volume between the surface* $z = x^3 \cos y$ *and the region* $1 \le x \le 2,\ \frac{\pi}{4} \le y \le \pi$.

$$\int_1^2 \int_{\pi/4}^{\pi} x^3 \cos y\ dy\ dx = \int_1^2 x^3 \sin y\Big|_{\pi/4}^{\pi} dx = \int_1^2 -\tfrac{\sqrt{2}}{2}x^3\ dx$$
$$= -\tfrac{\sqrt{2}}{8}x^4\Big|_1^2 = -2\sqrt{2} + \tfrac{\sqrt{2}}{8} = -\tfrac{15}{8}\sqrt{2}$$

Example 35.6. *Find the volume above the* xy*-plane between the cylinder* $x^2 + y^2 = 9$ *and the plane* $x+y+z=5$.

$$\iint_R (5-x-y)\ dy\ dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5-x-y)\ dy\ dx$$
$$= \int_{-3}^{3} (5y - xy - \tfrac{1}{2}y^2)\Big|_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\, dx$$
$$= \int_{-3}^{3} \left(10\sqrt{9-x^2} - 2x\sqrt{9-x^2}\right)\ dx$$
$$= 10(\tfrac{9\pi}{2}) = 45\pi$$

HOMEWORK FOR DAY 35. Page 234, #1 part a, #2

HOMEWORK ANSWERS. #1 a) Sketch the region to find that $0 \le x \le 1$, $0 \le y \le x$. Thus,

$$\int_0^1 \int_0^x (x^2 + y^2)\ dx\ dy = \int_0^1 \left(x^2 y + \tfrac{1}{3}y^3\right)\Big|_0^x dx = \int_0^1 \tfrac{4}{3}x^3\ dx = \tfrac{1}{3}x^4\Big|_0^1 = \tfrac{1}{3}$$

#2

a)

$$\int_0^1 \int_0^{\pi/2} e^x \cos y\ dy\ dx = \int_0^1 e^x \sin y\Big|_0^{\pi/2} dx = \int_0^1 e^x\ dx = e - 1$$

b)

$$\begin{aligned}\int_0^1 \int_0^2 x^2 e^{-x-y} dy\ dx &= \int_0^1 -x^2 e^{-x-y}\Big|_0^2 dx \\ &= \int_0^1 \left(-x^2 e^{-x-2} + x^2 e^{-x}\right)\ dx \\ &= (1 - e^{-2}) \int_0^1 x^2 e^{-x}\ dx\end{aligned}$$

We must use parts twice to find the antiderivative. We get

$$\begin{aligned}&= (1 - e^{-2})(-e^{-x})(x^2 + 2x + 2)\Big|_0^1 \\ &= (1 - e^{-2})(-5e^{-1} + 2)\end{aligned}$$

c)

$$\begin{aligned}\int_0^1 \int_{x+1}^{x+2} x^2 y\ dy\ dx &= \int_0^1 \tfrac{1}{2}x^2 y^2\Big|_{x+1}^{x+2}\ dx \\ &= \int_0^1 \left[\tfrac{1}{2}x^2(x^2 + 4x + 4) - \tfrac{1}{2}x^2(x^2 + 2x + 1)\right] dx \\ &= \int_0^1 (x^3 + \tfrac{3}{2}x^2)\ dx = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3\Big|_0^1 = \tfrac{3}{4}\end{aligned}$$

d) The condition $x^2 - y^2 \ge 0$ implies $-x \le y \le x$. Thus, we can evaluate by using a trigonometric substitution: $y = x\sin\theta$, so that $dy = x\cos\theta$.

$$\begin{aligned}\int_0^1 \int_{-x}^x \sqrt{x^2 - y^2}\ dy\ dx &= \int_0^1 \frac{x^2}{2}\left[\arcsin\frac{y}{x} + \frac{y}{x}\sqrt{x^2 - y^2}\right]\Bigg|_{-x}^x dx \\ &= \int_0^1 \frac{x^2}{2}[\pi]\ dx = \frac{\pi}{2}\left(\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{\pi}{6}\end{aligned}$$

36 Section 4.3 B, Applications of Double Integrals

Objective. *Students will investigate the properties of double integrals, including the Mean Value Theorem for Double Integrals. Students will apply double integrals to real-world situations, including mass, volume, and moments of inertia.*

First, we examine how to represent a given region R in terms of x or y; i.e., how to change the order of integration.

Example 36.1. *Change the order of integration of* $\int_0^2 \int_{x^2}^{2x} (4x+2)\ dy\ dx$.

Sketch the region R which consists of the region between $y = 2x$ and $y = x^2$ from $x = 0$ to $x = 2$. Then it is obvious that $0 \le y \le 4$ and $y/2 \le x \le \sqrt{y}$. Hence, the other order is $\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\ dx\ dy$.

Example 36.2. *Change the order of integration of* $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\ dx\ dy$.

The region is a semicircle on the half plane $y \ge 0$. Hence, this is equal to $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\ dy\ dx$.

Example 36.3. *Evaluate* $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y}\ dy\ dx$.

As given, we cannot integrate, since the integrand in y has no closed-form antiderivative. So we must change the order of integration: the region R is a triangle with vertices at the origin, $(0,\pi)$, and (π,π). Hence we have

$$\begin{aligned}\int_0^\pi \int_0^y \frac{\sin y}{y}\ dx\ dy &= \int_0^\pi \left.\frac{x \sin y}{y}\right|_0^y dy \\ &= \int_0^\pi \sin y\ dy = -\cos y|_0^\pi = 2\end{aligned}$$

Example 36.4. *Evaluate* $\displaystyle\int_1^e \int_0^{\log x} y\ dy\ dx$.

The region is in the first quadrant bounded by $y = \log x$ and $x = e$. Thus,

$$\begin{aligned}\int_0^1 \int_y^e y \, dx \, dy &= \int_0^1 xy\Big|_y^e dy \\ &= \int_0^1 (ey - y^2) \, dy = \tfrac{1}{2}ey^2 - \tfrac{1}{3}y^3\Big|_0^1 \\ &= \tfrac{1}{2}e - \tfrac{1}{3} = \tfrac{1}{6}(3e - 2)\end{aligned}$$

It is worthwhile to point out that the basic properties of single integrals hold for double integrals: sum of integrals, multiplication by a constant, etc. We also have that if R can be split into two disjoint regions R_1 and R_2, overlapping only at boundary points, then

$$\iint\limits_R f \, dx \, dy = \iint\limits_{R_1} f \, dx \, dy + \iint\limits_{R_2} f \, dx \, dy. \tag{36.1}$$

We also consider the following inequalities. If A is the area of R and $N_1 \le f \le N_2$ for points in R, we have

$$N_1 A \le \iint\limits_R f \, dx \, dy \le N_2 A. \tag{36.2}$$

Let $N = \max\{|N_1|, |N_2|\}$. Then,

$$\left| \iint\limits_R f \, dx \, dy \right| \le \iint\limits_R |f| \, dx \, dy \le NA. \tag{36.3}$$

Theorem 36.1 (The Mean Value Theorem for Double Integrals). *If f is continuous on the region R of area A, then for some $(x^*, y^*) \in R$ we have $\iint_R f(x,y) \, dx \, dy = f(x^*, y^*)A$. In other words, the average (mean) value of f over the region R is $\frac{1}{A} \iint_R f(x,y) \, dx \, dy$.*

Applications We list 5 applications of double integrals.

I) *Volume* If $f(x,y)$ is the equation of a surface, then $V = \iint_R f \, dx \, dy$ is the volume between the surface and the xy-plane.

II) *Area* For $f(x,y) = 1$, we have $A = \iint_R dx \, dy$ as the area of R.

III) *Mass* If $f(x,y)$ is the equation of the density of a surface (in mass per unit area), then $M = \iint_R f\ dx\ dy$ is the mass of R.

IV) *Center of Mass* If f is density, then the center of mass $(\overline{x}, \overline{y})$ of the thin plate represented by R is given by

$$\overline{x} = \frac{1}{M} \iint\limits_R xf\ dx\ dy, \qquad \overline{y} = \frac{1}{M} \iint\limits_R yf\ dx\ dy$$

where M is the mass of R.

V) *Moment of Inertia* The moment of inertia quantifies the resistance of a physical object to angular acceleration. Moment of inertia is to rotational motion as mass is to linear motion. An object's moment of inertia depends on its shape and the distribution of mass within that shape: the greater the concentration of material away from the object's center, the larger the moment of inertia. If f is density, and R the thin plate, the moments of inertia about the x-axis and y-axis are

$$I_x = \iint\limits_R y^2 f\ dx\ dy, \qquad I_y = \iint\limits_R x^2 f\ dx\ dy,$$

and the polar moment of inertia (inertia about the origin) is $I_O = I_x + I_y$.

Example 36.5. *Find the center of mass and the moments of inertia of the thin plate covering the triangular region with vertices* $(0,0)$, $(1,0)$, *and* $(1,2)$, *given that the plate's density is* $f(x,y) = 12x + 12y + 6$.

First, we find M, the mass of the plate:

$$\int_0^1 \int_0^{2x} (12x + 12y + 6)\ dy\ dx = \int_0^1 (12xy + 6y^2 + 6y)\Big|_0^{2x} dx$$
$$= \int_0^1 (48x^2 + 12x)\ dx = 16x^3 + 6x^2\Big|_0^1 = 22.$$

Now we have

$$\overline{x} = \tfrac{1}{22} \int_0^1 \int_0^{2x} (12x^2 + 12xy + 6x)\ dy\ dx$$
$$= \tfrac{1}{22} \int_0^1 (12x^2 y + 6xy^2 + 6xy)\Big|_0^{2x} dx = \tfrac{1}{22} \int_0^1 (48x^3 + 12x^2)\ dx$$
$$= \tfrac{1}{22}\,(12x^4 + 4x^3)\Big|_0^1 = \tfrac{1}{22}(16) = \tfrac{8}{11}$$

Similarly, we find $\overline{y} = \frac{9}{11}$; hence, the center of mass is at the point $\left(\frac{8}{11}, \frac{9}{11}\right)$. The moment of inertia about the x-axis is given by

$$\begin{aligned} I_x &= \int_0^1 \int_0^{2x} (12xy^2 + 12y^3 + 6y^2)\ dy\ dx \\ &= \int_0^1 (4xy^3 + 3y^4 + 2y^3)\Big|_0^{2x} = \int_0^1 (80x^4 + 16x^3)\ dx \\ &= 16x^5 + 4x^4\Big|_0^1 = 20 \end{aligned}$$

Similarly, the moment of inertia about the y-axis is $\frac{68}{5}$; hence the moment of inertia about the origin is $\frac{68}{5} + 20 = \frac{168}{5}$.

HOMEWORK FOR DAY 36. Page 235, #1 part c, #3 part a, #5; Verify $\overline{y}$ and I_y from Example 36.5

HOMEWORK ANSWERS. #3 a) The region is a trapezoid with vertices $(1, 0)$, $(2, -1)$, $(2, 3)$, and $(1, 2)$; there are three limits for x, and we have no choice but to use a piecewise function: the integral is

$$\int_{-1}^{3} \int_{h(y)}^{2} f(x, y)\; dx\; dy, \text{ where } h(y) = \begin{cases} 1 - y & -1 \leq y \leq 0 \\ 1 & 0 \leq y \leq 2 \\ y - 1 & 2 \leq y \leq 3 \end{cases}$$

#5

a) R is a triangle with vertices $(1/2, 0)$, $(1, 0)$, and $(1/2, 1/2)$; thus, the integral is $\int_0^{1/2} \int_{1/2}^{1-y} f\; dx\; dy$.

b) R is quartercircle of radius 1 in the first quadrant; thus, the integral is $\int_0^1 \int_0^{\sqrt{1-y^2}} f\; dx\; dy$.

c) R is a triangle with vertices $(-1, 0)$, $(0, 0)$, and $(0, 1)$; thus, the integral is $\int_{-1}^0 \int_0^{x+1} f\; dy\; dx$.

d) R is again a triangle with vertices $(1, 0)$, $(1, 2)$, and $(0, 1)$; thus, R can be split along the line $y = 1$ into two congruent pieces, so that the integral is then $2 \int_0^1 \int_{1-y}^1 f\; dx\; dy$.

Verifying $\overline{y}$ and I_y from Example 36.5 is easy.

37 Sections 4.4 and 4.5, Triple Integrals and Integrals of Vector Functions

Objective. *Students will evaluate triple integrals and integrals of vector functions.*

The definition is

$$\iiint_R f(x,y,z)\;dx\;dy\;dz = \lim_{h\to 0}\sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\Delta_i V$$

where $\Delta_i V$ is the volume of the ith rectangular parallelepiped in R, h is the maximum space diagonal of the n parallelepipeds, and (x_i^*, y_i^*, z_i^*) is an arbitrary point in the ith parallelepiped. —[[Larson 110]]—

Reduction to an iterated integral given the region R defined by $a \le x \le b,\ g_1(x) \le y \le g_2(x),\ z_1(x,y) \le z \le z_2(x,y)$ is

$$\iiint_R f(x,y,z)\;dx\;dy\;dz = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\;dz\;dy\;dx.$$

Since a single integral determines area and a double integral determines volume, it is natural to assume that a triple integral determines some type of "hypervolume." But it is simpler to interpret this as mass: $\iiint_R f\;dx\;dy\;dz$ is the mass of a solid of density f, and $\iiint_R dx\;dy\;dz$ is the volume of the region R. The center of mass and moments of inertia are defined similarly; the Mean Value Theorem holds as well.

Example 37.1. *Evaluate $\iiint_R (xy^2 + yz^3)\;dx\;dy\;dz$ if R is defined by $-1 \le x \le 1,\ 3 \le y \le 4,\ 0 \le z \le 2$.*

$$\begin{aligned}
\int_3^4 \int_{-1}^1 \int_0^2 (xy^2 + yz^3)\;dz\;dx\;dy \\
&= \int_3^4 \int_{-1}^1 \left(xy^2 z + \tfrac{1}{4}yz^4\right)\Big|_0^2 dx\;dy \\
&= \int_3^4 \int_{-1}^1 (2xy^2 + 4y)\;dx\;dy = \int_3^4 x^2y^2 + 4xy\Big|_{-1}^1 dy \\
&= \int_3^4 8y\;dy = 4y^2\Big|_3^4 = 28
\end{aligned}$$

Example 37.2. *Evaluate* $\iiint_R (2x - y - z)\ dz\ dy\ dx$ *if* R *is defined by* $0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le x + y.$

$$\begin{aligned}
&\int_0^1 \int_0^{x^2} \int_0^{x+y} (2x - y - z)\ dz\ dy\ dx \\
&\qquad = \int_0^1 \int_0^{x^2} \left(2xz - yz - \frac{1}{2}z^2\right)\Big|_0^{x+y} dy\ dx \\
&\qquad = \int_0^1 \int_0^{x^2} \frac{3}{2}(x^2 - y^2)\ dy\ dx = \frac{3}{2}\int_0^1 \left(x^2 y - \frac{1}{3}y^3\right)\Big|_0^{x^2} dx \\
&\qquad = \frac{3}{2}\int_0^1 \left(x^4 - \frac{1}{3}x^6\right)\ dx = \frac{3}{2}\left(\frac{1}{5}x^5 - \frac{1}{21}x^7\right)\Big|_0^1 = \frac{8}{35}
\end{aligned}$$

Example 37.3. *Evaluate* $\iiint_R (x + z)\ dz\ dy\ dx$ *if* R *is the tetrahedron with vertices* $A(0,0,0)$*,* $B(1,0,0)$*,* $C(0,2,0)$*, and* $D(0,0,3)$*.*

Clearly $0 \le x \le 1$. In the xy-plane, the tetrehedron forms a right triangle whose hypotenuse is the line $y = 2 - 2x$; hence, $0 \le y \le 2 - 2x$. To determine the height of a cross section of the tetrahedron, it becomes necessary to determine the plane including the non-axial face of the tetrahedron. Thus,

$$\overrightarrow{BD} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 0 & 3 \\ -1 & 2 & 0 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

is a vector perpendicular to the plane. Using point B, the equation of the plane is

$$-6(x - 1) - 3y - 2z = 0,$$

or $2z = -6x - 3y + 6$. Hence, $0 \le z \le -3x - \frac{3}{2}y + 3$. Therefore, the integral is

$$\iiint\limits_R (x+z)\ dz\ dy\ dx$$

$$= \int_0^1 \int_0^{2-2x} \int_0^{-3x-3y/2+3} (x+z)\ dz\ dy\ dx$$

$$= \int_0^1 \int_0^{2-2x} \left(xz + \frac{1}{2}z^2 \right)\Big|_0^{-3x-3y/2+3} dy\ dx$$

$$= \int_0^1 \int_0^{2-2x} \left(\frac{9}{2} - 6x + \frac{3}{2}x^2 + 3xy - \frac{9}{2}y + \frac{9}{8}y^2 \right)\ dy\ dx$$

$$= \int_0^1 \left(\frac{9}{2}y - 6xy + \frac{3}{2}x^2 y + \frac{3}{2}xy^2 - \frac{9}{4}y^2 + \frac{3}{8}y^3 \right)\Big|_0^{2-2x} dx$$

$$= \int_0^1 3\left(1 - 2x + x^2\right)\ dx = 3\left(x - x^2 + \frac{1}{3}x^3 \right)\Big|_0^1 = 1$$

Example 37.4. *Evaluate* $\iiint_R f(x,y,z)\ dV$ *for* $f(x,y,z) = x^2 + z^2$ *and* R *as the pyramid with vertices* $(\pm 1, \pm 1, 0)$ *and* $(0,0,1)$*.*

Here, we consider only an eighth of the pyramid as the region, and multiply the integral by 8. The region under consideration is then a tetrahedron with vertices at the origin, $(0,0,1)$, $(1,1,0)$ and $(1,0,0)$. We have $0 \le x \le 1$ and $0 \le y \le x$. By the methods in the previous example, the non-axial face is the plane $x + z = 1$ so that $0 \leq z \leq 1 - x$. Therefore,

$$\begin{aligned}
\int_0^1 \int_0^x \int_0^{1-x} &(x^2 + z^2)\ dz\ dy\ dx \\
&= \int_0^1 \int_0^x \left(x^2 z + \frac{1}{3} z^3 \right) \Big|_0^{1-x} dy\ dx \\
&= \int_0^1 \int_0^x \left[x^2(1-x) + \frac{1}{3}(1-x)^3 \right]\ dy\ dx \\
&= \int_0^1 \left[x^2(1-x) + \frac{1}{3}(1-x)^3 \right] y \Big|_0^x dx \\
&= \int_0^1 x \left[x^2(1-x) + \frac{1}{3}(1-x)^3 \right]\ dx \\
&= \int_0^1 \left(-\frac{4}{3}x^4 + 2x^3 - x^2 + \frac{1}{3}x \right)\ dx \\
&= -\frac{4}{15}x^5 + \frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{1}{6}x^2 \Big|_0^1 \\
&= -\frac{4}{15} + \frac{1}{2} - \frac{1}{3} + \frac{1}{6} = \frac{1}{15}
\end{aligned}$$

Hence, $\iiint_R (x^2 + z^2)\ dV = \frac{8}{15}$.

Vector Fucntions Integrals of vector function follow normal rules, see page 234.

HOMEWORK FOR DAY 37. Page 234, #1 part b)—approximate final intergal numerically; Page 235, #4 part a; Page 236, #10

HOMEWORK ANSWERS. #1 b) The condition $u^2+v^2 \leq 1$ implies $-1 \leq u \leq 1$ and $-\sqrt{1-u^2} \leq v \leq \sqrt{1-u^2}$. Thus, the integral becomes

$$\begin{aligned}
\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{0}^{1} u^2v^2w \; dw \; dv \; du &= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{1}{2}u^2v^2w^2\Big|_0^1 \, dv \; dw \\
&= \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \frac{1}{2}u^2v^2 \; dv \; du \\
&= \int_{-1}^{1} \frac{1}{6}u^2v^3\Big|_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du \\
&= \int_{-1}^{1} \frac{1}{3}u^2(1-u^2)\sqrt{1-u^2} \; du \\
&= \frac{\pi}{48} \approx 0.065
\end{aligned}$$

#4 a) Since the cube given is the unit cube, we have the following integral:

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 \sqrt{x+y+z} \; dz \; dy \; dx \\
&= \int_0^1\int_0^1 \frac{2}{3}(x+y+z)^{3/2}\Big|_0^1 dy \; dx \\
&= \int_0^1\int_0^1 \frac{2}{3}\left((x+y+1)^{3/2} - (x+y)^{3/2}\right) \; dy \; dx \\
&= \int_0^1 \frac{2}{3}\left(\frac{2}{5}(x+y+1)^{5/2} - \frac{2}{5}(x+y)^{5/2}\right)\Big|_0^1 dx \\
&= \int_0^1 \frac{4}{15}\left((x+2)^{5/2} - 2(x+1)^{5/2} + x^{5/2}\right) \; dx \\
&= \frac{4}{15}\left(\frac{2}{7}(x+2)^{7/2} - \frac{4}{7}(x+1)^{7/2} + \frac{2}{7}x^{7/2}\right)\Big|_0^1 \\
&= \frac{8}{105}\left(3^{7/2} - 2(2^{7/2}) + 1 - 2^{7/2} + 2\right) \\
&= \frac{8}{105}\left(27\sqrt{3} - 24\sqrt{2} + 3\right) \\
&= \frac{8}{35}\left(9\sqrt{3} - 8\sqrt{2} + 1\right)
\end{aligned}$$

#10 a)

$$\int_0^1 \mathbf{F}(t)\ dt = \tfrac{1}{3}t^3\mathbf{i} - e^t\mathbf{j} + \log(1+t)\mathbf{k}\Big|_0^1$$

$$= \tfrac{1}{3}\mathbf{i} - e\mathbf{j} + (\log 2)\mathbf{k} + \mathbf{j} = \tfrac{1}{3}\mathbf{i} + (1-e)\mathbf{j} + (\log 2)\mathbf{k}$$

#10 b)

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \mathbf{F}(t)\ dt &= \int_0^1 \left(\frac{1}{2}x^2y^2\mathbf{i} + \frac{1}{3}xy^3\mathbf{j}\right)\Bigg|_0^{1-x} dx \\
&= \int_0^1 \left[\frac{1}{2}(x^2 - 2x^3 + x^4)\mathbf{i} + \frac{1}{3}(x - 3x^2 + 3x^3 - x^4)\mathbf{j}\right] dx \\
&= \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\mathbf{i} + \frac{1}{3}\left(\frac{1}{2}x^2 - x^3 + \frac{3}{4}x^4 - \frac{1}{5}x^5\right)\mathbf{j}\Bigg|_0^1 \\
&= \frac{1}{60}(\mathbf{i} + \mathbf{j})
\end{aligned}$$

38 Section 4.6, Change of Variables in Double Integrals

Objective. *Students will prove the substitution method of integration. Students will change variables in double integrals to evaluate them.*

We review, and prove, substitution in integrals of single variable functions.

Theorem 38.1. *If $f(x)$ is continuous on $[a, b]$, where $x = x(u)$ is defined on $[a_u, b_u]$ and has a continuous derivative, with $a = x(a_u)$, $b = x(b_u)$, and $f(x(u))$ is continuous on $[a_u, b_u]$, then $\int_a^b f(x)\, dx = \int_{a_u}^{b_u} f(x(u))\frac{dx}{du}du$.*

Proof. Let $F(x)$ be an antiderivative of $f(x)$. Then $\int_a^b f(x)\, dx = F(b)-F(a)$. Next, by the Chain Rule, we have

$$\frac{dF}{du} = \frac{df}{dx}\frac{dx}{du} = f(x)\frac{dx}{du} = f(x(u))\frac{dx}{du},$$

so that $F(x(u))$ is an antiderivative of $f(x(u))\frac{dx}{du}$. Thus,

$$\int_{a_u}^{b_u} f(x(u))\frac{dx}{du}du = F(x(b_u)) - F(x(a_u)) = F(b) - F(a).$$

□

Example 38.1. *Evaluate* $\displaystyle\int_0^{\pi/4} \frac{x\cos x(x\sin x - \cos x)}{1 + x\cos x}\, dx.$

Let $t = 1 + x\cos x$. Then $dt = (\cos x - x\sin x)\, dx$. Note that $t = 1$ when $x = 0$ and that $t = 1 + \frac{\pi\sqrt{2}}{8}$ when $x = \frac{\pi}{4}$. Then the integral becomes

$$\begin{aligned}
\int_1^{1+\pi\sqrt{2}/8} -\frac{t-1}{t}\, dt &= \int_1^{1+\pi\sqrt{2}/8} \left(1 - \frac{1}{t}\right)\, dt \\
&= -(t - \log t)\big|_1^{1+\pi\sqrt{2}/8} \\
&= -\left(1 + \frac{\pi\sqrt{2}}{8} - \log\left(1 + \frac{\pi\sqrt{2}}{8}\right) - 1\right) \\
&= \log\left(1 + \frac{\pi\sqrt{2}}{8}\right) - \frac{\pi\sqrt{2}}{8}
\end{aligned}$$

Substitution in double integrals has the form

$$\iint\limits_{R_{xy}} f(x,y)\ dx\ dy = \iint\limits_{R_{uv}} f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\ dv.$$

Note that we use the absolute value of the Jacobian.

Example 38.2. *Reduce* $\iint_{R_{xy}} f(x,y)\ dx\ dy$ *to an iterated integral of polar functions* $x = r\cos\theta$, $y = r\sin\theta$, *where* R_{xy} *is the triangle with vertices* $(0,0)$, $(1,0)$, *and* $(1,1)$.

First, we find the Jacobian:

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Clearly, the region must be described in terms of r and θ. We have $0 \le x \le 1,\ 0 \le y \le x$. The y interval becomes $0 \le \theta \le \frac{\pi}{4}$. The x interval becomes $0 \le r \le \sec\theta$. Hence,

$$\iint_{R_{xy}} f(x,y)\ dx\ dy = \int_0^{\pi/4}\int_0^{\sec\theta} f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta.$$

Example 38.3. *Evaluate* $\iint_{R_{xy}} e^{-x^2-y^2}\ dx\ dy$ *where* R_{xy} *is the region in the first quadrant bounded by* $x^2+y^2=1$ *and* $x^2+y^2=4$.

We convert this to polar. The region in the polar plane is bounded by $r = 1$ and $r = 2$; since the region is the first quadrant, we have $0 \le \theta \le \frac{\pi}{2}$. From the previous example, the Jacobian is r, so

$$\begin{aligned}\iint_{R_{xy}} e^{-x^2-y^2}\ dx\ dy &= \iint_{R_{r\theta}} e^{-r^2} r\ dr\ d\theta = \int_0^{\pi/2}\int_1^2 e^{-r^2} r\ dr\ d\theta \\ &= \int_0^{\pi/2} -\frac{1}{2}e^{-r^2}\Big|_1^2\ d\theta = \int_0^{\pi/2} -\frac{1}{2}(e^{-4}-e^{-1})\ d\theta \\ &= -\frac{\pi}{4}(e^{-4}-e^{-1}) \approx 0.275\end{aligned}$$

Example 38.4. *A region in the xy-plane is a parallelogram with vertices* $(0,0)$, $(4,2)$, $(5,5)$ *and* $(1,3)$. *Find a transformation to the uv-plane so the region becomes a rectangle.*

Answer: $u = \frac{1}{2}x - y$, $v = 3x - y$ so that $-\frac{5}{2} \le u \le 0$ and $0 \le v \le 10$.

Example 38.5. *Evaluate* $\iint_{R_{xy}} (x+y)^3 \, dx \, dy$ *where* R_{xy} *is the parallelogram with sides* $x + y = 1$, $x + y = 4$, $x - 2y = 1$, *and* $x - 2y = -2$.

We introduce a substitution defined by $x + y = u$ and $x - 2y = v$ which becomes the rectangle $1 \le u \le 4$, $-2 \le v \le 1$ in the uv-plane. Thus, since

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = -\frac{1}{3},$$

we have

$$\iint\limits_{R_{xy}} (x+y)^3 \, dx \, dy = \int_{-2}^{1} \int_{1}^{4} \frac{u^3}{3} \, du \, dv = \frac{255}{4}.$$

Example 38.6. *Evaluate* $\iint_R e^{(y-x)/(y+x)} \, dx \, dy$ *where* R *is the trapezoid in the first quadrant bounded by* $x + y = 1$ *and* $x + y = 2$.

We use $u = y - x$ and $v = y + x$. Clearly, $1 \le v \le 2$. We find bounds on u (which correspond to the axial sides of the trapezoid) by solving the system

$$\begin{cases} y - x = u \\ y + x = v \end{cases}$$

two ways: for x to get $2y = u + v$, in which case $u = -v$ along the line $y = 0$; and for y to get $2x = v - u$, in which case $u = v$ along the line $x = 0$. Hence, $-v \le u \le v$. Next,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = -\frac{1}{2}$$

so the integral is

$$\begin{aligned} \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{u/v} \, du \, dv &= \frac{1}{2} \int_1^2 v e^{u/v} \Big|_{-v}^{v} dv = \frac{1}{2} \int_1^2 v(e - e^{-1}) \, dv \\ &= \frac{1}{4}(e - e^{-1}) \, v^2 \Big|_1^2 = \frac{3}{4}(e - e^{-1}) \approx 1.763 \end{aligned}$$

Example 38.7. *Evaluate* $\iint_R \frac{(x-y)^2}{1+x+y} \, dx \, dy$ *where* R *is the trapezoid bounded by* $x + y = 1$ *and* $x + y = 2$ *in the first quadrant.*

We use $u = 1 + x + y$ and $v = x - y$. Clearly, $2 \le u \le 3$. We find bounds on v (which correspond to the axial sides of the trapezoid) by solving the system

$$\begin{cases} x + y = u - 1 \\ x - y = v \end{cases}$$

two ways: for x to get $2x = u - 1 + v$, in which case $v = x$ along the line $y = 0$ so that $v = u - 1$; and for y to get $2y = u - 1 - v$, in which case $v = -y$ along the line $x = 0$ so that $v = 1 - u$. Hence, $1 - u \le v \le u - 1$. Next,

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}$$

so the integral is

$$\begin{aligned} \int_2^3 \int_{1-u}^{u-1} \frac{v^2}{2u}\, dv\, du &= \int_2^3 \left. \frac{v^3}{6u} \right|_{1-u}^{u-1} du \\ &= \int_2^3 \left(\frac{u^2}{3} - u + 1 - \frac{1}{3u} \right) du \\ &= \left. \frac{u^3}{9} - \frac{u^2}{2} + u - \frac{\log u}{3} \right|_2^3 \\ &= 3 - \frac{9}{2} + 3 - \frac{\log 3}{3} - \frac{8}{9} + 2 - 2 + \frac{\log 2}{3} \\ &= \frac{11}{18} - \frac{\log(3/2)}{3} \end{aligned}$$

HOMEWORK FOR DAY 38. Page 241, #1 parts a and b (part c is extra credit), #4 parts a, b, and c

HOMEWORK ANSWERS. #1

a) Since $x = \sin\theta$, $dx = \cos\theta d\theta$. $x \in [0,1]$ implies $\theta \in [0, \frac{\pi}{2}]$. Hence, the integral is

$$\begin{aligned}\int_0^{\pi/2} (1-\sin^2\theta)^{3/2}\cos\theta\ d\theta &= \int_0^{\pi/2} \cos^4\theta d\theta \\ &= \int_0^{\pi/2} \left[\cos^2\theta - (\cos\theta\sin\theta)^2\right]\ d\theta \\ &= \int_0^{\pi/2} \left[\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right] d\theta \\ &= \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\Big|_0^{\pi/2} = \frac{3\pi}{16}\end{aligned}$$

b) Since $x = u^2 - 1$, $dx = 2u\ du$. $x \in [0,1]$ implies $u \in [1, \sqrt{2}]$. Then the integral is

$$\int_1^{\sqrt{2}} \frac{2u}{1+u}\ du = \int_2^{1+\sqrt{2}} \frac{2(v-1)}{v}\ dv$$

where we have another substitution $v = 1+u$, where $v \in [2, 1+\sqrt{2}]$.

$$\begin{aligned}&= 2\int_2^{1+\sqrt{2}} \left(1 - \frac{1}{v}\right)\ dv = 2(v - \log v)\big|_2^{1+\sqrt{2}} \\ &= 2\left(1+\sqrt{2} - \log(1+\sqrt{2}) - 2 + \log 2\right) \\ &= 2\sqrt{2} - 2 + \log(2\sqrt{2} - 2)\end{aligned}$$

c) Since $t = \tan\frac{x}{2}$, we have $\sin\frac{x}{2} = \frac{t}{\sqrt{t^2+1}}$ and $\cos\frac{x}{2} = \frac{1}{\sqrt{t^2+1}}$. Therefore, since $dt = \frac{1}{2}\sec^2\frac{x}{2}\ dx$, we have $dx = 2\cos^2\frac{x}{2}\ dt = \frac{2}{t^2+1}\ dt$. Next, we need all angles in terms of x rather than $\frac{x}{2}$; thus, using double-angle identities, we get $\sin x = \frac{2t}{t^2+1}$ and $\cos x = \frac{1-t^2}{t^2+1}$. Thus, the integral becomes

$$\begin{aligned}
\int_0^1 \frac{\frac{2}{t^2+1}\ dt}{\frac{2t}{t^2+1} + \frac{1-t^2}{t^2+1} + 2} &= \int_0^1 \frac{2\ dt}{t^2+2t+3} = \int_0^1 \frac{2\ dt}{(t+1)^2+2} \\
&= \int_0^1 \frac{dt}{\left(\frac{t+1}{\sqrt{2}}\right)^2 + 1} = \sqrt{2}\arctan\left(\frac{t+1}{\sqrt{2}}\right)\Big|_0^1 \\
&= \sqrt{2}\left(\arctan\sqrt{2} - \arctan\frac{1}{\sqrt{2}}\right)
\end{aligned}$$

#4

a) The condition $x^2 + y^2 \leq 1$ implies $r^2 \leq 1$, so $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Ergo, the integral is

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} (1-r^2)r\ d\theta\ dr &= \int_0^1 \left(r\theta - r^3\theta\right)\Big|_0^{2\pi} \\
&= \int_0^1 (2\pi r - 2\pi r^3)\ dr = 2\pi\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}
\end{aligned}$$

b) Since $0 \leq y \leq x$, $0 \leq \theta \leq \frac{\pi}{4}$. Also, $1 \leq x \leq 2$ implies that $\sec\theta \leq r \leq 2\sec\theta$. Hence,

$$\begin{aligned}
\int_0^{\pi/4} \int_{\sec\theta}^{2\sec\theta} r^2 \tan\theta\ dr\ d\theta &= \int_0^{\pi/4} \frac{1}{3} r^2 \tan\theta \Big|_{\sec\theta}^{2\sec\theta} d\theta \\
&= \frac{1}{3}\int_0^{\pi/4} (7\sec^2\theta\tan\theta)\ d\theta \\
&= \frac{7}{6}\sec^2\theta\Big|_0^{\pi/4} = \frac{7}{2}
\end{aligned}$$

c) We have a square bounded by the lines $y - x = \pm\pi$ and $y + x = \pi, y + x = 3\pi$. Then $-\pi \leq u \leq \pi$ and $\pi \leq v \leq 3\pi$. Next,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{1}{2},$$

so the integral is

$$\begin{aligned}\frac{1}{2}\int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2\sin^2 v\ dv\ du &= \frac{1}{2}\int_{-\pi}^{\pi} u^2\left(\frac{1}{2}v - \frac{1}{4}\sin 2v\right)\Big|_{\pi}^{3\pi} du \\ &= \frac{1}{2}\int_{-\pi}^{\pi} u^2\left(\frac{3\pi}{2} - \frac{\pi}{2}\right)\ du \\ &= \frac{\pi}{6}u^3\Big|_{-\pi}^{\pi} = \frac{\pi^4}{3}\end{aligned}$$

39 Section 4.7, Arc Length and Surface Area

Objective. *Students will find arc length of space curves. Students will prove the formula for surface area of a surface in space and apply it.*

Length of a curve in space defined parametrically has length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

The derivation of the surface area of a surface in space has much to do with arc length of a curve in a plane. We review this.

Let $f(x)$ be defined on $[a,b]$. For some $c_i \in [x_i, x_i + h]$ we find the tangent to $f(x)$ at c_i to be $y - y(c_i) = f'(c_i)(x - c_i)$. Let T_h be the length of the tangent from x_i to $x_i + h$. Then

$$L = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{i=1}^{n} T_h.$$

Let α_i be the angle T_h makes with the x-axis. Then $\cos\alpha_i = h/T_h$, or $T_h = h\sec\alpha_i$. Then

$$\begin{aligned}\lim_{\substack{n\to\infty \\ h\to 0}} \sum_{i=1}^{n} T_h &= \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{i=1}^{n} h\sec\alpha_i = \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{i=1}^{n} h\sqrt{1+\tan^2\alpha_i} \\ &= \lim_{\substack{n\to\infty \\ h\to 0}} \sum_{i=1}^{n} h\sqrt{1+[f'(x_i)]^2} = \int_a^b \sqrt{1+[f'(x)]^2}\, dx\end{aligned}$$

Theorem 39.1. *If $z = f(x,y)$ is defined and has continuous partial derivatives in $R \subseteq D$, then the surface area of a surface in space is*

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\ dy.$$

Proof. Let (x_i, y_i) be a point of the ith rectangle of the subdivision of $R \subseteq D$. Then the tangent plane at (x_i, y_i) is

$$z - z_i = \frac{\partial z}{\partial x}(x - x_i) + \frac{\partial z}{\partial y}(y - y_i)$$

where the partials are evaluated at (x_i, y_i). Let S_i be the area of the part of tangent plane above the ith rectangle. Then S_i is the area of a parallelogram whose projection is a rectangle on R with area A_i. Let $\mathbf{n}_i$ be the normal to z at (x_i, y_i); hence, —[[THOMAS 13.38, 13.39, 13.40]]—

$$\mathbf{n}_i = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}.$$

Then $S_i = \sec\gamma_i A_i$, where γ_i is the angle between $\mathbf{n}_i$ and $\mathbf{k}$. Note that

$$\cos\gamma_i = \frac{\mathbf{n}_i \cdot \mathbf{k}}{||\mathbf{n}_i||||\mathbf{k}||} = \frac{1}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}$$

so that

$$\sec\gamma_i = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

Hence, if we let d_i denote the diagonal of the ith rectangle,

$$\begin{aligned}
\lim_{\substack{n\to\infty \\ d\to 0}} \sum_{i=1}^{n} S_i &= \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{i=1}^{n} A_i \sec\gamma_i \\
&= \lim_{\substack{n\to\infty \\ d\to 0}} \sum_{i=1}^{n} A_i \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \\
&= \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy
\end{aligned}$$

□

Example 39.1. *Find the surface area of the paraboloid $z = x^2 + y^2$ bounded by $x^2 + y^2 = 4$.*

—[[THOMAS 13.41]]— We have that $-\sqrt{4 - x^2} \le y \le \sqrt{4 - x^2}$ and $-2 \le x \le 2$ Thus,

$$A = \iint_R \sqrt{1 + (2x)^2 + (2y)^2}\, dx\, dy = \iint_R \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy.$$

This integral is tedious; we change to polar. Then $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Hence, since the Jacobian is r,

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^2 r\sqrt{4r^2+1}\ dr\ d\theta = \int_0^{2\pi} \frac{1}{12}(4r^2+1)^{3/2}\Big|_0^2\ d\theta \\ &= \int_0^{2\pi} \frac{1}{12}(17^{3/2}-1)\ d\theta = \frac{\pi}{6}(17^{3/2}-1) \\ &\approx 36.177 \end{aligned}$$

Theorem 39.2. *Let $x = x(u,v),\ y = y(u,v),\ z = z(u,v)$ be the parametrization of a surface in space. If all first partials are defined in $R \subseteq D$, then the surface area is*

$$S = \iint_{R_{uv}} \sqrt{EG - F^2}\ du\ dv$$

where

$$\begin{aligned} E &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \\ F &= \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \\ G &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2. \end{aligned}$$

Example 39.2. *Consider a torus parametrized by $x = (2+\cos v)\cos u,\ y = (2+\cos v)\sin u,\ z = \sin v$ for $0 \le u, v \le 2\pi$. Find its surface area.*

We compute E, F, and G.

$$\begin{aligned} E &= (2+\cos v)^2\sin^2 u + (2+\cos v)^2\cos^2 u = (2+\cos v)^2 \\ F &= (2+\cos v)\sin u \sin v \cos u - (2+\cos v)\cos u \sin v \sin u = 0 \\ G &= \sin^2 v\cos^2 u + \sin^2 v \sin^2 u + \cos^2 v = 1 \\ EG - F^2 &= (2+\cos v)^2 \end{aligned}$$

Hence, the surface area is

$$S = \int_0^{2\pi}\int_0^{2\pi} (2+\cos v)\ dv\ du = \int_0^{2\pi} (2v + \sin v)\Big|_0^{2\pi} du = \int_0^{2\pi} 4\pi\ du = 8\pi^2.$$

HOMEWORK FOR DAY 39. Page 248, #2 part b, #3; Page 249 #9 (#2 part a is extra credit)

HOMEWORK ANSWERS. #2 a) We use the positive root of z to compute the surface area of the hemisphere; then multuiply the result by 2. Since R is the circle $x^2 + y^2 = a^2$ so $-a \le x \le a$, $-\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2}$. Note that $\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2-x^2-y^2}}$ and $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2-x^2-y^2}}$. Thus,

$$
\begin{aligned}
\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} & \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}}\; dy\; dx \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\; dy\; dx \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{\frac{a}{\sqrt{a^2 - x^2}}}{\sqrt{1 - \frac{y^2}{a^2 - x^2}}}\; dy\; dx \\
&= \int_{-a}^{a} a \arcsin \frac{y}{\sqrt{a^2 - x^2}}\Bigg|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx \\
&= \int_{-a}^{a} a\pi\; dx = a\pi x|_{-a}^{a} = 2a^2\pi.
\end{aligned}
$$

Thus, the surface area of the entire sphere is $4a^2\pi$.

#2 b) We compute E, F, and G.

$$
\begin{aligned}
E &= a^2\cos^2\phi\cos^2\theta + a^2\cos^2\phi\sin^2\theta + a^2\sin^2\phi = a^2 \\
F &= -a^2\cos\phi\cos\theta\sin\phi\sin\theta + a^2\cos\phi\sin\theta\sin\phi\cos\theta = 0 \\
G &= a^2\sin^2\phi\sin^2\theta + a^2\sin^2\phi\cos^2\theta = a^2\sin^2\phi \\
EG - F^2 &= a^4\sin^2\phi
\end{aligned}
$$

Hence, the surface area is

$$
\begin{aligned}
S = \int_0^{2\pi}\int_0^{\pi}\sqrt{a^4\sin^2\phi}\; d\phi\; d\theta &= a^2\int_0^{2\pi}\int_0^{\pi}\sin\phi\; d\phi\; d\theta \\
&= a^2\int_0^{2\pi} -\cos\phi|_0^{\pi}\, d\theta \\
&= 2a^2\theta\Big|_0^{2\pi} = 4a^2\pi
\end{aligned}
$$

#3 a) We compute E, F, and G.

$$\begin{aligned}
E &= (b + a\cos v)^2 \sin^2 u + (b + a\cos v)^2 \cos^2 u = (b + a\cos v)^2 \\
F &= a(b + a\cos v)\sin u \sin v \cos u - a(b + a\cos v)\cos u \sin v \sin u = 0 \\
G &= a^2 \sin^2 v \cos^2 u + a^2 \sin^2 v \sin^2 u + a^2 \cos^2 v = a^2 \\
EG - F^2 &= a^2(b + a\cos v)^2
\end{aligned}$$

Hence, the surface area is

$$\begin{aligned}
S = \int_0^{2\pi} \int_0^{2\pi} a(b + a\cos v)\; dv\; du &= a \int_0^{2\pi} (bv + a\sin v)\big|_0^{2\pi}\, du \\
&= a \int_0^{2\pi} 2\pi b\; du \\
&= 2\pi abu\big|_0^{2\pi} = 4\pi^2 ab
\end{aligned}$$

#3 b) We compute E, F, and G.

$$\begin{aligned}
E &= 4u^2 \sin^2 2v + 1 \\
F &= 4u^3 \sin 2v \cos 2v \\
G &= u^2 + 4u^4 \cos^2 2v \\
EG - F^2 &= u^2 + 4u^4
\end{aligned}$$

Hence, the surface area is

$$\begin{aligned}
S = \int_0^1 \int_0^{\pi/2} u\sqrt{1 + 4u^2}\; dv\; du &= \frac{\pi}{2} \int_0^1 u\sqrt{1 + 4u^2}\; du \\
&= \frac{\pi}{16} \cdot \frac{2}{3}(1 + 4u^2)^{3/2}\Big|_0^1 \\
&= \frac{\pi}{24}(5^{3/2} - 1)
\end{aligned}$$

#9 Use Equation 2.59 on page 106 to rewrite the integrand, then find a common denominator.

40 Section 4.8, Improper Multiple Integrals

Objective. *Students will use a limit process to evaluate improper double integrals.*

Review single-variable improper integrals. Distinguish between bounded and unbounded.

Discontinuities could be points or boundaries; i.e., $\sin \frac{y}{x}$ is discontinuous on the y-axis, and $\frac{1}{x+y}$ is discontinuous only at the origin.

Example 40.1. *Evaluate* $\iint_R \frac{1}{\sqrt{x^2+y^2}^p}\ dx\ dy$ *where* $R = \{(x,y)|x^2+y^2 \leq 1\}$.

We have a point discontinuity at the origin. We use polar to integrate:

$$\begin{aligned}\int_0^{2\pi}\int_h^1 \frac{1}{r^p}\ r\ dr\ d\theta &= \int_0^{2\pi}\int_h^1 \frac{1}{r^{p-1}}\ dr\ d\theta = \int_0^{2\pi} \left.\frac{r^{2-p}}{2-p}\right|_h^1 d\theta \\ &= \int_0^{2\pi}\left(\frac{1-h^{2-p}}{2-p}\right)\ d\theta = 2\pi\left(\frac{1-h^{2-p}}{2-p}\right)\end{aligned}$$

As $h \to 0$, we get $\frac{2\pi}{2-p}$ if $p < 2$. If $p > 2$ we get $\frac{2\pi}{2-p}\left(1 - \frac{1}{h^{p-2}}\right)$ and as $h \to 0$, the integral diverges. If $p = 2$,

$$\begin{aligned}\int_0^{2\pi}\int_h^1 \frac{1}{r}\ dr\ d\theta &= \int_0^{2\pi} \log r|_h^1\, d\theta \\ &= \int_0^{2\pi} -\log h\ d\theta = -2\pi \log h\end{aligned}$$

and as $h \to 0$, it diverges. *Note the similarity with single variable at* $p = 1$.

Example 40.2. *Evaluate* $\iint_R -\log xy\ dx\ dy$ *if* R *is the square* $0 < x, y \leq 1$.

$$\begin{aligned}\int_h^1\int_h^1 -\log xy\ dx\ dy &= \int_h^1\int_h^1 (-\log x - \log y)\ dx\ dy \\ &= \int_h^1 (-x\log x + x - x\log y)|_h^1\, dy \\ &= \int_h^1 (1 - \log y + h\log h - h + h\log y)\ dy \\ &= 2y - y\log y + hy\log h + hy\log y - 2hy|_h^1 \\ &= 2 + 2h\log h - 4h - 2h^2\log h + 2h^2\end{aligned}$$

As $h \to 0$, we get 2.

Example 40.3. *Evaluate* $\iint_R 1/x^2y^2\ dx\ dy$ *where* R *is the infinite region* $x \geq 1,\ y \geq 1$.

$$\int_1^h \int_1^h \frac{1}{x^2y^2}\ dx\ dy = \int_1^h \left.\frac{-1}{xy^2}\right|_1^h dy = \int_1^h \left(-\frac{1}{hy^2} + \frac{1}{y^2}\right)\ dy$$
$$= \left.\frac{1}{hy} - \frac{1}{y}\right|_1^h = \frac{1}{h^2} - \frac{2}{h} + 1$$

As $h \to \infty$, we get 1.

Example 40.4. *Evaluate* $\iint_R \log\sqrt{x^2+y^2}\ dx\ dy$ *where* R *is the unit disk* $x^2 + y^2 \leq 1$.

We use polar to integrate.

$$\int_0^{2\pi} \int_h^1 r \log r\ dr\ d\theta = \int_0^{2\pi} \left.\frac{r^2}{4}(2\log r - 1)\right|_h^1 d\theta$$
$$= \int_0^{2\pi} \left(-\frac{1}{4} - \frac{h^2}{4}(2\log h - 1)\right)\ d\theta$$
$$= -\frac{\pi}{2} - \frac{\pi h^2}{2}(2\log h - 1)$$

As $h \to 0$, we get $-\pi/2$.

HOMEWORK FOR DAY 40. Page 252 #1; Page 253 #4 parts b and c

HOMEWORK ANSWERS. #1 Using polar to evaluate the equations, we have that the region $x \geq 0,\ y \geq 0$ becomes $r \geq 0,\ 0 \leq \theta \leq \pi/2$. Thus,

$$\begin{aligned}\int_0^{\pi/2}\int_0^h e^{-r^2} r\ dr\ d\theta &= \int_0^{\pi/2} -\frac{e^{-r^2}}{2}\bigg|_0^h d\theta \\ &= \int_0^{\pi/2} \frac{1-e^{-h^2}}{2}\ d\theta \\ &= \frac{\pi(1-e^{-h^2})}{4}\end{aligned}$$

As $h \to \infty$ we get $\pi/4$; hence the original integral is $\sqrt{\pi/4} = \frac{1}{2}\sqrt{\pi}$.

#4 b)

$$\begin{aligned}\int_0^{2\pi}\int_h^1 \frac{\log r^2}{r} r\ dr\ d\theta &= 2\int_0^{2\pi}\int_h^1 \log r\ dr\ d\theta \\ &= 2\int_0^{2\pi} (r\log r - r)|_h^1\, d\theta \\ &= 2\int_0^{2\pi} (-1 - h\log h + h)\ d\theta \\ &= 4\pi(h - 1 - h\log h) + 1\end{aligned}$$

As $h \to 0$, we have $1 - 4\pi$.

#4 c)

$$\begin{aligned}\int_0^{2\pi}\int_1^h 2r\log r\ dr\ d\theta &= 2\int_0^{2\pi} \frac{r^2}{4}(2\log r - 1)\bigg|_1^h\ d\theta \\ &= 2\int_0^{2\pi}\left(\frac{h^2}{4}(2\log h - 1) + \frac{1}{4}\right)\ d\theta \\ &= 4\pi\left(\frac{h^2}{4}(2\log h - 1) + \frac{1}{4}\right)\end{aligned}$$

As $h \to \infty$, the integral diverges.

41 Section 4.9, Integrals Depending on a Parameter

An integral of a function $f(x,t)$ such as $\int_a^b \sin(xt)\ dx = F(t)$ is a function of the parameter t. For example, $\int_0^\pi \sin(xt)\ dx = \frac{1-\cos(\pi t)}{t} = F(t)$.

We can tabulate this for values of t, and we get a well-defined function. If $F(t) = \int_a^b f(x,t)\ dx$ has no antiderivative, we still get a well-defined function. But then we focus on the derivative of $F(t)$.

Theorem 41.1 (Leibniz's Rule). *Let $f(x,t)$ be continuous and have a continuous derivative $\partial f/\partial t$ in a domain of the xt-plane that includes the rectangle $a \le x \le b,\ t_1 \le t \le t_2$. Then for $t \in [t_1,t_2]$, we have*

$$\frac{d}{dt}\int_a^b f(x,t)\ dx = \int_a^b \frac{\partial f}{\partial t}(x,t)\ dx.$$

Proof. For $t \in [t_1,t_2]$, let

$$g(t) = \int_a^b \frac{\partial f}{\partial t}(x,t)\ dx.$$

Since $\partial f/\partial t$ is continuous, so is $g(t)$, by Fubini's Theorem. Let $t^* \in [t_1,t_2]$. Then

$$\begin{aligned}\int_{t_1}^{t^*} g(t)\ dt &= \int_{t_1}^{t^*}\int_a^b \frac{\partial f}{\partial t}(x,t)\ dx\ dt = \int_a^b\int_{t_1}^{t^*} \frac{\partial f}{\partial t}(x,t)\ dt\ dx \\ &= \int_a^b \left[f(x,t^*) - f(x,t_1)\right]\ dx = \int_a^b f(x,t^*)\ dx - \int_a^b f(x,t_1)\ dx \\ &= F(t^*) - F(t_1).\end{aligned}$$

Now, if we let t^* be a variable t, we have $F(t) - F(t_1) = \int_{t_1}^t g(u)\ du$. Both sides can be differentiated with respect to t to get $F'(t) = g(t)$. □

Example 41.1. *Evaluate* $\dfrac{d}{dt}\displaystyle\int_0^{\pi/2} \frac{\sin(xt)}{x}\ dx.$

$$\begin{aligned}\frac{d}{dt}\int_0^{\pi/2} \frac{\sin(xt)}{x}\ dx &= \int_0^{\pi/2} \cos(xt)\ dx = \frac{\sin(xt)}{t}\bigg|_0^{\pi/2} \\ &= \frac{\sin(\pi t/2)}{t}\end{aligned}$$

Theorem 41.2. *Let $f(x,t)$ be continuous and have a continuous derivative $\partial f/\partial t$ in a domain of the xt-plane. Let $a(t)$ and $b(t)$ be defined and have continuous derivatives for $t_1 \le t \le t_2$. Then for $t \in [t_1, t_2]$,*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\;dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)\;dx.$$

Example 41.2. *Evaluate* $\dfrac{d}{dt}\displaystyle\int_{t}^{\tan t} x^2 t\;dx.$

$$\begin{aligned}\frac{d}{dt}\int_{t}^{\tan t} x^2 t\;dx &= t\tan^2 t\sec^2 t - t^3 + \int_{t}^{\tan t} x^2\;dx \\ &= t\tan^2 t\sec^2 t - t^3 + \frac{1}{3}\tan^3 t - \frac{1}{3}t^3\end{aligned}$$

HOMEWORK FOR DAY 41. Page 256, #1 parts a, b, and c, #2 parts a, b, and c, #4 part a

42 Section 5.2, Line Integrals

Objective. *Students will evaluate line integrals in the plane.*

A *smooth curve* C is a curve of the form $x = \phi(t),\ y = \psi(t)$ for $h \le t \le k$, where x and y are continuous and have continuous derivatives.

C can have a direction, usually in increasing values of t.

Let $A = (\phi(t), \psi(t)),\ B = (\phi(k), \psi(k))$ so C can be a path from A to B. If $A = B$, the C is a *closed curve*; if all points on C are distinct and $A = B$, then C is a *simple closed curve.*

If $f(x, y)$ is defined (at least) when (x, y) is on C, and if f represents density or force, then the line integral

$$\int_C f(x,y)\ ds = \lim \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta_i s$$

is the mass of the wire C, or the work done to move a particle along C. (See page 271.) Above, s is the the parameter of length along the curve.

In practical terms, we separate x and y and compute

$$\int_C f(x,y)\ dx \quad \text{and} \quad \int_C f(x,y)\ dy \tag{42.1}$$

where the first integral is the mass of or work done by the x-component; similarly for the second integral.

Theorem 42.1. *If $f(x, y)$ is continuous on C, then the integrals in Eq. 42.1 both exist and can be defined parametrically by*

$$\int_C f(x,y)\ dx = \int_h^k f(\phi(t), \psi(t))\phi'(t)\ dt$$
$$\int_C f(x,y)\ dy = \int_h^k f(\phi(t), \psi(t))\psi'(t)\ dt$$

The theorem implies that line integrals can be reduced to regular integrals. C may not be smooth, but all results still hold as long as C is piecewise smooth. Further, any parametrization of C may be used.

Example 42.1. *Let C be the path defined by $y = (x-1)^2$ from $(1,0)$ to $(2,1)$. Evaluate $\int_C (x^2 - y^2)\ dx$ and $\int_C (x^2 - y^2)\ dy$.*

We choose the parametrization $x = t,\ y = (t-1)^2$ for $1 \le t \le 2$. Then

$$\int_C (x^2 - y^2)\ dx = \int_1^2 \left[t^2 - (t-1)^4\right]\ dt = \frac{32}{15}$$

$$\int_C (x^2 - y^2)\ dy = \int_1^2 2(t-1)\left[t^2 - (t-1)^4\right]\ dt = \frac{5}{2}$$

If we choose a different parametrization—say, $x = t+1,\ y = t^2$ for $0 \le t \le 1$—then

$$\int_C (x^2 - y^2)\ dx = \int_0^1 \left[(t+1)^2 - t^4\right]\ dt = \frac{32}{15}$$

$$\int_C (x^2 - y^2)\ dy = \int_0^1 2t\left[(t+1)^2 - t^4\right]\ dt = \frac{5}{2}.$$

Note that if C is given by $y = g(x)$ for $a \le x \le b$, then we may use x as the "parameter." Then we have

$$\int_C f(x,y)\ dx = \int_a^b f(x, g(x))\ dx$$

$$\int_C f(x,y)\ dy = \int_a^b f(x, g(x))g'(x)\ dx$$

If C is given by $x = G(y)$ for $c \le y \le d$, then we have similar equations.

Example 42.2. *Evaluate $\int_C (x^2 - y^3)\ dy$ where C is the semicircle $y = \sqrt{1-x^2}$ from $(1,0)$ to $(-1,0)$.*

We choose the parametrization $x = \cos t,\ y = \sin t$ for $0 \le t \le \pi$. Then the integral becomes

$$\begin{aligned}
\int_0^\pi (\cos^3 t - \sin^3 t)\cos t \; dt &= \int_0^\pi \left[\cos^2 t - (\cos t \sin t)^2 - \sin^3 t \cos t\right] \; dt \\
&= \int_0^\pi \left[\frac{1}{2} + \frac{\cos 2t}{2} - \frac{\sin^2 2t}{4} - \sin^3 t \cos t\right] \; dt \\
&= \frac{t}{2} + \frac{\sin 2t}{4} - \frac{t}{8} + \frac{\sin 4t}{32} - \frac{\sin^4 t}{4}\Bigg|_0^\pi \\
&= \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8}
\end{aligned}$$

If we had used x as the parameter, the integral becomes

$$\int_{-1}^{1} \left[x^3 - (1-x^2)^{3/2}\right] \frac{-x}{\sqrt{1-x^2}} \; dx$$

which is clearly more difficult to evaluate.

In many situations, different functions determine x and y; i.e., $f(x, y)$ could be given by a vector function $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and the line integral is then denoted by

$$\int\limits_C P(x, y) \; dx + Q(x, y) \; dy.$$

Example 42.3. *Evaluate* $\int\limits_C xy^2 \; dx + x^2 y \; dy$ *if* C *is the arc* $y = x^2$ *from* $(0, 0)$ *to* $(-1, 1)$.

We have

$$\int\limits_C xy^2 \; dx + x^2 y \; dy = \int_0^{-1} \left(xy^2 + x^2 y \frac{dy}{dx}\right) \; dx = \int_0^{-1} 3x^5 \; dx = \frac{1}{2}.$$

If C is a closed curve, a direction must be specified and we use the notation

$$\oint\limits_C P \; dx + Q \; dy.$$

Example 42.4. *Evaluate* $\oint\limits_C y^2 \; dx + x^2 \; dy$ *where* C *is the triangle with vertices* $(1, 0), (1, 1), (0, 0)$.

We compute three integrals; one for each side of the triangle. The first is from $(0,0)$ to $(1,0)$. Along this path, $y = 0$ and $dy = 0$, so $y^2\ dx + x^2\ dy = 0$ which gives the first integral as zero. The second is from $(1,0)$ to $(1,1)$. Along this leg, $x = 1$ and $dx = 0$ so $y^2\ dx + x^2\ dy = dy$. Hence, $\int_0^1 dy = 1$. The third integral is from $(1,1)$ to $(0,0)$. Along this path, $x = y$ and $dx = dy$, so that $y^2\ dx + x^2\ dy = 2x^2\ dx$ and we have $\int_1^0 2x^2\ dx = -2/3$. Hence,

$$\oint_C y^2\ dx + x^2\ dy = 0 + 1 - \frac{2}{3} = \frac{1}{3}.$$

Example 42.5. *Evaluate $\int_{(2,1)}^{(4,5)} xy\ dx + x^2\ dy$ along a) the broken line from $(2,1)$ to $(4,1)$ to $(4,5)$; b) the straight line from $(2,1)$ to $(4,5)$; and c) the curve defined by $x = 3t - 1,\ y = 3t^2 - 2t$ for $1 \le t \le \frac{5}{3}$.*

a) From $(2,1)$ to $(4,1)$, we have $y = 1$ and $dy = 0$. Then the integral along this part is $\int_2^4 x\ dx = 6$. From $(4,1)$ to $(4,5)$, we have $x = 4$ and $dx = 0$. Then the integral along this part is $\int_1^5 16\ dy = 64$. Hence,

$$\int_{(2,1)}^{(4,5)} xy\ dx + x^2\ dy = 6 + 64 = 70.$$

b) The line from $(2,1)$ to $(4,5)$ is $y = 2x - 3$. Then, since $dy = 2\ dx$, we have

$$\int_{(2,1)}^{(4,5)} xy\ dx + x^2\ dy = \int_2^4 [x(2x-3) + 2x^2]\ dx = \int_2^4 (4x^2 - 3x)\ dx = \frac{170}{3}.$$

c) We have $dx = 3\ dt$ and $dy = (6t - 2)\ dt$. Accordingly,

$$\begin{aligned}\int_{(2,1)}^{(4,5)} xy\ dx + x^2\ dy &= \int_1^{5/3} [3(3t-1)(3t^2-2t) + (3t-1)^2(6t-2)]\ dt \\ &= \int_1^{5/3} (81t^3 - 81t^2 + 24t - 2)\ dt \\ &= \frac{81}{4}t^4 - 27t^3 + 12t^2 - 2t\Big|_1^{5/3} \\ &= \frac{625}{4} - 125 + \frac{100}{3} - \frac{10}{3} - \frac{81}{4} + 27 - 12 + 2 = 58\end{aligned}$$

HOMEWORK FOR DAY 42. Page 278 #1, #2 parts a and b; Page 279 #3 parts a and c

HOMEWORK ANSWERS. #2 a) We use y as the parameter.

$$\begin{aligned}\int_{(0,-1)}^{(0,1)} y^2\ dx + x^2\ dy &= \int_{-1}^{1} \left(y^2\frac{dx}{dy} + x^2\right)\ dy \\ &= \int_{-1}^{1} \left(\frac{-y^3}{\sqrt{1-y^2}} + 1 - y^2\right)\ dy\end{aligned}$$

Using parts on the rational term, with $u = y^2$ and $dv = -y/\sqrt{1-y^2}$, gives

$$\begin{aligned}&= \frac{1}{3}\left((y^2+2)\sqrt{1-y^2} - y^3 + 3y\right)\Big|_{-1}^{1} \\ &= \frac{1}{3}(-1+3-1+3) = \frac{4}{3}\end{aligned}$$

#2 b) We use x as the parameter.

$$\int_{(0,0)}^{(2,4)} y\ dx + x\ dy = \int_0^2 \left(y + x\frac{dy}{dx}\right)\ dx = \int_0^2 3x^2\ dx = 8$$

#3 a) Along $(-1,-1)$ to $(1,-1)$, $y=-1$ and $dy=0$. Then $\int_{-1}^{1} dx = 2$. Along $(1,-1)$ to $(1,1)$, $x=1$ and $dx=0$; thus, $\int_{-1}^{1} y\ dy = 0$. Along $(1,1)$ to $(-1,1)$, $y=1$ and $dy=0$. Then $\int_{1}^{-1} dx = -2$. Along the last leg, we have $x=-1$ and $dx=0$, so $\int_{1}^{-1} -y\ dy = 0$. The total is 0.

#3 b) Along $(0,0)$ to $(1,0)$, $y = dy = 0$ and the integral is then 0. Along $(1,0)$ to $(1,1)$, $x=1$ and $dx=0$; then $\int_0^1 -y^3\ dy = -1/4$. Along $(1,1)$ to $(0,0)$, $y=x$ and $dx=dy$ so the integral is 0. The total is $-1/4$.

43 Section 5.3, Integrals with Respect to Arc Length

Objective. *Students will compute line intgerals with respect to arc length. Students will understand and apply basic properties of line integrals.*

The basic line integral is arc length $s = \int_h^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int\limits_C ds$ which represents the distance traveled on C.

If f is continuous on C, then

$$\int\limits_C f(x,y)\ dx = \int_h^k f(\phi(t), \psi(t))\sqrt{\phi'(t)^2 + \psi'(t)^2}\ dt$$

is the integral of f with respect to arc length s. If s itself is the parameter, then $x = x(s),\ y = y(s)$ and

$$\int\limits_C f(x,y)\ ds = \int_0^L f(x(s), y(s))\ ds.$$

If x is the parameter, then $y = y(x)$ and

$$\int\limits_C f(x,y)\ ds = \int_a^b f(x, y(x))\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx.$$

The basic line integral $\int P\ dx + Q\ dy$ becomes $\int\limits_C (P\cos\alpha + Q\sin\alpha)\ ds$ where α is the angle between the x-axis and a tangent vector in the direction of increasing s.

Example 43.1. *Evaluate* $\int_C xy^2\ ds$ *where* C *is the curve* $x = \cos t,\ y = \sin t$ *for* $0 \le t \le \frac{\pi}{2}$.

$$\int\limits_C xy^2\ ds = \int_0^{\pi/2} \cos t \sin^2 t\sqrt{\sin^2 t + \cos^2 t}\ dt = \int_0^{\pi/2} \cos t \sin^2 t\ dt$$

$$= \tfrac{1}{3}\sin^3 t\Big|_0^{\pi/2} = \tfrac{1}{3}$$

Example 43.2. *Find* $\int_{(0,0)}^{(2,4)} (x+y)\, ds$ *where* C *is the curve* $y = x^2$.

We use x as the parameter.

$$\begin{aligned}
&\int_0^2 (x+x^2)\sqrt{1+4x^2}\, dx \\
&\quad = \frac{1}{8}\int_0^2 8x\sqrt{1+4x^2}\, dx + 2\int_0^2 x^2\sqrt{\tfrac{1}{4}+x^2}\, dx \\
&\quad = \frac{1}{12}(1+4x^2)^{3/2} + \frac{1}{4}\left[x\left(\tfrac{1}{4}+2x^2\right)\sqrt{\tfrac{1}{4}+x^2} - \frac{1}{16}\log\left(x+\sqrt{\tfrac{1}{4}+x^2}\right)\right]\Bigg|_0^2 \\
&\quad = \frac{1}{12}(17^{3/2}-1) + \frac{1}{4}\left[\frac{33\sqrt{17}}{4} - \frac{1}{16}\log\left(2+\frac{\sqrt{17}}{2}\right) + \frac{1}{16}\log\frac{1}{2}\right] \\
&\quad = \frac{167}{48}\sqrt{17} - \frac{1}{16}\log\left(4+\sqrt{17}\right) - \frac{1}{12} \\
&\quad \approx 14.229
\end{aligned}$$

(The antiderivative of the second integral is found by using a combination of parts and trigonometric substitution.)

Example 43.3. *Find* $\oint_C 2xy\, ds$ *if* C *is the upper semicircle* $x^2+y^2=9$.

We use t as the parameter by setting $x = 3\cos t,\ y = 3\sin t$ for $0 \le t \le \pi$. Then

$$\begin{aligned}
&\int_0^\pi 18\sin t\cos t\sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\
&\quad = 27\int_0^\pi 2\sin t\cos t\, dt = 27\int_0^\pi \sin 2t\, dt \\
&\quad = -\frac{27}{2}\cos 2t\Big|_0^\pi = 27
\end{aligned}$$

Properties of Line Integrals

Additivity of portions of curves; additivity of integrands; reversal of endpoints; constant multiplication.

We also have the following:

$$\int_C ds = L = \text{length of } C \tag{43.1}$$

If $|f(x,y)| \leq M$ on C, then

$$\left| \int\limits_C f(x,y)\ ds \right| \leq ML \tag{43.2}$$

If C is a simple closed curve, then

$$\oint\limits_C x\ dy = -\oint\limits_C y\ dx = \text{area enclosed by } C. \tag{43.3}$$

Example 43.4. *Evaluate $\int_C ds$ if C is the straight line from the origin to $(3,3)$.*

This is simply the length of C; by the Pythagorean Theorem, this length is $3\sqrt{2}$.

Example 43.5. *Evaluate $\oint_C x\ dy$ if C is the ellipse defined by $x = 3\cos t,\ y = 5\sin t$.*

By Eq. 43.3, this is simply the area of the region enclosed by C; the area of the ellipse is 15π.

HOMEWORK FOR DAY 43. Page 279 #3 part b, #4

HOMEWORK ANSWERS. #3 b) By Eq. 43.3, we have that the integral is $2\oint_C y\ dx = -2\pi$.

#4 a) We use the parametrization $x = 2\cos t,\ y = 2\sin t$ for $0 \le t \le \pi$.

$$\begin{aligned}\oint_C (x^2 - y^2)\ ds &= \int_0^{2\pi} (4\cos^2 t - 4\sin^2 t)\sqrt{4\sin^2 t + 4\cos^2 t}\ dt \\ &= 8\int_0^{2\pi} (\cos^2 t - \sin^2 t)\ dt = 8\int_0^{2\pi} \cos 2t\ dt \\ &= 4\sin 2t\Big|_0^{2\pi} = 0\end{aligned}$$

#4 b) From $(0,0)$ to $(1,1)$, the length of $y = x$ is $\sqrt{2}$. Then

$$\int_C x\ ds = \int_0^1 x\sqrt{2}\ dx = \frac{\sqrt{2}}{2}.$$

#4 c) This is just standard arc length; thus,

$$\int_C ds = \int_0^1 \sqrt{1+4x^2}\ dx$$

Let $x = \frac{1}{2}\tan t$ to get

$$\begin{aligned}&= \frac{1}{2}\int_0^{\arctan 2} \sqrt{1+\tan^2 t}\sec^2 t\ dt \\ &= \frac{1}{2}\int_0^{\arctan 2} \sec^3 t\ dt\end{aligned}$$

By parts, we have

$$\begin{aligned}&= \frac{1}{4}\left[\log|\sec t + \tan t| + \sec t\tan t\right]\Big|_0^{\arctan 2} \\ &= \frac{1}{4}\log\left(\sqrt{5}+2\right) + \frac{\sqrt{5}}{2} \\ &\approx 1.479\end{aligned}$$

44 Section 5.4, Line Integrals as Integrals of Vectors

Objective. *Students will evaluate line integrals of vector-valued functions.*

If $P(x, y)$ and $Q(x, y)$ are components of the vector $\mathbf{u} = P\mathbf{i} + Q\mathbf{j}$, then we define u_T as the component of $\mathbf{u}$ in the direction of the unit tangent $\mathbf{T}$:

$$u_T = \mathbf{u} \cdot \mathbf{T} = P\frac{dx}{ds} + Q\frac{dy}{ds},$$

so then

$$\int\limits_C u_T\ ds = \int\limits_C P\ dx + Q\ dy.$$

We can also define it directly by using the differential vector $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$:

$$\int\limits_C u_T\ ds = \int\limits_C P\ dx + Q\ dy = \int\limits_C \mathbf{u} \cdot d\mathbf{r}.$$

If C is represented parametrically in terms of t, then

$$\int\limits_C \mathbf{u} \cdot d\mathbf{r} = \int_h^k \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right) dt = \int_h^k \left(\mathbf{u} \cdot \frac{d\mathbf{r}}{dt}\right)\ dt.$$

Application to Mechanics

If $\mathbf{u} = P\mathbf{i} + Q\mathbf{j}$ is a force field, then the line integral represents the work done by this force in moving the particle along C. Further, if $\mathbf{r}$ is the position vector of a particle of mass m moving on C and $\mathbf{u}$ is the force applied, then

$$\mathbf{u} = m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d\mathbf{v}}{dt}$$

by Newton's Second Law, where $\mathbf{v}$ is the velocity of the particle along C. If we denote $||\mathbf{v}|| = v$, then

$$\int\limits_C \mathbf{u} \cdot d\mathbf{r} = \int_h^k \left(\mathbf{u} \cdot \frac{d\mathbf{r}}{dt}\right)\ dt = \int_h^k \left(m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\right)\ dt$$

$$= \int_h^k \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}\right)\ dt = \int_h^k \frac{d}{dt}\left(\frac{1}{2}mv^2\right)\ dt = \frac{1}{2}mv^2\Big|_h^k;$$

in other words, *the work done equals the gain in kinetic energy.*

Example 44.1. *Evaluate* $\int_C v_T \, ds$ *if* $\mathbf{v} = x^2y\mathbf{i} + (y-x)\mathbf{j}$ *and if* C *is the curve* $y = (x-1)^2$ *from* $(0,1)$ *to* $(1,0)$.

We use t as the parameter for the representation $x = t+1, \ y = t^2$ for $-1 \le t \le 0$.

$$\begin{aligned}
\int_C v_T \, ds &= \int_C x^2y \, dx + (y-x) \, dy \\
&= \int_{-1}^{0} \left[(t+1)^2t^2 + (t^2 - t - 1)(2t)\right] \, dt \\
&= \int_{-1}^{0} (t^4 + 4t^3 - t^2 - 2t) \, dt \\
&= \frac{t^5}{5} + t^4 - \frac{t^3}{3} - t^2 \Big|_{-1}^{0} \\
&= \frac{1}{5} - 1 - \frac{1}{3} + 1 = -\frac{2}{15}
\end{aligned}$$

HOMEWORK FOR DAY 44. Page 286 #1

HOMEWORK ANSWERS. #1

a)

$$\int_C v_T\ ds = \int_0^1 (x^2 + x^2)\ dx + 2xx\ dy$$
$$= \int_0^1 4x^2\ dx = \frac{4}{3}x^3\bigg|_0^1 = \frac{4}{3}$$

b)

$$\int_C v_T\ ds = \int_0^1 (x^2 + x^4)\ dx + 2xx^2\ dy$$
$$= \int_0^1 (5x^4 + x^2)\ dx = x^5 + \frac{1}{3}x^3\bigg|_0^1 = \frac{4}{3}$$

c) Along $(0,0)$ to $(1,0)$, $y = dy = 0$; so $\int_0^1 x^2\ dx = \frac{1}{3}$. Along $(1,0)$ to $(1,1)$, $x = 1$ and $dx = 0$; so $\int_0^1 2y\ dy = 1$. Thus, $\int_C v_T\ ds = \frac{4}{3}$.

45 Section 5.5, Green's Theorem

Objective. *Students will evaluate line integrals of closed curves by applying Green's Theorem.*

Theorem 45.1 (Green's Theorem)**.** *Let D be a domain of the xy-plane and let C be a piecewise smooth closed curve in D, with interior also in D. Let $P(x,y)$ and $Q(x,y)$ be functions defined and continuous with continuous first partial derivatives in D. Then*

$$\oint_C P(x,y)\ dx + Q(x,y)\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy.$$

Proof. We prove the case where R can be represented as

$$a \le x \le b,\ f_1(x) \le y \le f_2(x)$$

or

$$c \le y \le d,\ g_1(y) \le x \le g_2(y).$$

—[[Larson 113]]— We can write:

$$\begin{aligned}
\oint_R \frac{\partial P}{\partial y}\ dx\ dy &= \int_a^b \int_{f_1}^{f_2} \frac{\partial P}{\partial y}\ dy\ dx \\
&= \int_a^b \left[P(x,f_2) - P(x,f_1)\right]\ dx \\
&= -\int_b^a P(x,f_2)\ dx - \int_a^b P(x,f_1)\ dx \\
&= -\oint_C P(x,y)\ dx
\end{aligned}$$

Similarly,

$$\iint_R \frac{\partial Q}{\partial x}\ dx\ dy = \oint_C Q(x,y)\ dx\ dy.$$

Hence,

$$\oint_C P(x,y)\ dx + Q(x,y)\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy.$$

□

If R can be decomposed into finitely many such regions R_i, with C_i as the corresponding boundaries, Green's Theorem still holds. Any other type of region can be approximated by such regions and a limiting process employed.

Example 45.1. *Let C be the circle $x^2+y^2=1$. Evaluate $\oint_C xy\ dx+(y^2-x)\ dy$.*

We have

$$\begin{aligned}
\oint\limits_C xy\ dx+(y^2-x)\ dy &= \iint\limits_R (-1-x)\ dx\ dy \\
&= -\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x+1)\ dy\ dx \\
&= -\int_{-1}^{1} 2(x+1)\sqrt{1-x^2}\ dx \\
&= -2\left[-\frac{1}{3}(1-x^2)^{3/2}\right]_{-1}^{1} - 2\left(\frac{\pi}{2}\right) \\
&= -\pi
\end{aligned}$$

Example 45.2. *Let C be the circle $x^2+y^2=r^2$. Evaluate $\oint_C (2x+y)\ dx+(x+3y)\ dy$.*

We have

$$\oint\limits_C (2x+y)\ dx+(x+3y)\ dy = \iint_R (1-1)\ dx\ dy = 0.$$

Example 45.3. *Let C be the circle $x^2+y^2=r^2$. Evaluate $\oint_C (2x-y)\ dx+(x+3y)\ dy$.*

We have

$$\begin{aligned}
\oint\limits_C (2x-y)\ dx+(x+3y)\ dy &= \iint\limits_R (1+1)\ dx\ dy \\
&= 2\iint\limits_R dx\ dy = 2\pi r^2
\end{aligned}$$

since the double integral is the area of R.

So when $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ is constant, the line integral is a multiple of the area of R.

Example 45.4. *Let C be a curve in the xy-plane. Then*

$$\oint_C \left(\frac{1}{xy}\ dx + \frac{x}{x^2+y^2}\ dy \right)$$

cannot be evaluated using Green's Theorem since P and Q are discontinuous at the origin.

Example 45.5. *Let C be the unit square. Evalute $\oint_C (x^2-y^2)\ dx + (x+y)\ dy$.*

We have

$$\begin{aligned}\oint_C (x^2-y^2)\ dx + (x+y)\ dy &= \iint_R (1+2y)\ dx\ dy \\ &= \iint_R dx\ dy + 2\iint_R y\ dx\ dy \\ &= 1 + 2\overline{y} = 1 + 2(\tfrac{1}{2}) = 2\end{aligned}$$

Vector Interpretation of Green's Theorem

We know that if $\mathbf{u} = P\mathbf{i} + Q\mathbf{j}$, then $\oint_C P\ dx + Q\ dy = \oint_C u_T\ ds$. Notice that the z-component of $\text{curl}_z\ \mathbf{u} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$, so we can write $\oint_C P\ dx + Q\ dy = \iint_R \text{curl}_z\ \mathbf{u}\ dx\ dy$.

This can also be interpreted differently. Let $\mathbf{v} = Q\mathbf{i} - P\mathbf{j}$ and $\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$ be the outer normal on C. Then

$$\oint_C P\ dx + Q\ dy = \oint_C \mathbf{v}\cdot\mathbf{n}\ ds = \oint_C v_{\mathbf{n}}\ ds = \iint_R \text{div}\,\mathbf{v}\ dx\ dy.$$

Example 45.6. *Let $\mathbf{v} = (x+y^2)\mathbf{i} - (x^3-y)\mathbf{j}$ and C be the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Evaluate $\oint_C v_{\mathbf{n}}\ ds$.*

We have

$$\begin{aligned}\oint_C v_{\mathbf{n}}\ ds &= \iint_R (1+1)\ dx\ dy \\ &= 2\iint_R dx\ dy = 2\cdot 2\cdot 3\cdot \pi = 12\pi\end{aligned}$$

HOMEWORK FOR DAY 45. Page 287 #5 parts a, b, c, d, e, and h

HOMEWORK ANSWERS. #5

a) $\oint_C ay\ dx + bx\ dy = \iint_R (b-a)\ dx\ dy = (b-a)A$ where A is the area of R.

b) From $(0,0)$ to $(1,0)$, $y = dy = 0$; so the integral becomes 0. From $(1,0)$ to $(1,\pi/2)$, $x = 1$, $dx = 0$; so $\oint_C e\cos y\ dy = 0$. From $(1,\pi/2)$ to $(0,\pi/2)$, $y = \pi/2$, $dy = 0$; so $\oint_C e^x\ dx = 0$. From $(0,\pi/2)$ to $(0,0)$, $x = dx = 0$; so $\oint_C \cos y\ dy = 0$. Hence the total integral us 0.

c) By Green's Theorem, the integral becomes

$$\begin{aligned}
\iint\limits_R (3x^2+3y^2)\ dx\ dy &= 3\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\ dy\ dx \\
&= \int_{-1}^{1} (3x^2y+y^3)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} 2(2x^2+1)\sqrt{1-x^2}\ dx \\
&= \int_{-1}^{1} 4x^2\sqrt{1-x^2}\ dx + 2\int_{-1}^{1}\sqrt{1-x^2}\ dx
\end{aligned}$$

Using the substitution $x = \sin\theta$, the first integrand becomes $\sin^2 2\theta\ d\theta$ from $\theta = 0$ to $\theta = \pi$. The second integral is the area of a semi-circle. Hence,

$$\begin{aligned}
&= \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\Big|_0^{\pi} + 2\left(\frac{\pi}{2}\right) \\
&= \frac{\pi}{2} + \pi = \frac{3}{2}\pi
\end{aligned}$$

d) $\mathbf{u} = \nabla(x^2y) = 2xy\mathbf{i} + x^2\mathbf{j}$. So $\oint_C u_T\ ds = \iint_R (2x-2x)\ dx\ dy = 0$.

e) $\oint_C v_{\mathbf{n}}\ ds = \iint_R (2x-2x)\ dx\ dy = 0$.

h) $\oint_C f(x)\ dx + g(y)\ dy = 0$.

46 Section 5.6A, Independence of Path

Objective. *Students will evaluate line integrals that are independent of path by evaluating the exact differential. Students will identify exact differentials and conservative vector fields.*

Example 46.1. *Evaluate $\int_{(0,0)}^{(1,1)} y\ dx + (x+2y)\ dy$ where C is a) the broken line from $(0,0)$ to $(2,1)$ to $(1,1)$; b) the parabolic arc $y = x^2$; and c) the straight line $y = x$.*

a) Along the line from $(0,0)$ to $(2,1)$, we have $y = \frac{1}{2}x$ so that $dy = \frac{1}{2}dx$. Then this part is $\int_0^2 [\frac{1}{2}x + \frac{1}{2}(x+x)]\ dx = \frac{3}{4}x^2|_0^2 = 3$. Along the line from $(2,1)$ to $(1,1)$, we have $y = 1$ and $dy = 0$. Then this part is $\int_2^1 dx = x|_2^1 = -1$. Hence,

$$\int_{(0,0)}^{(1,1)} y\ dx + (x+2y)\ dy = 3 + (-1) = 2.$$

b) Since $dy = 2x\ dx$, we have

$$\int_{(0,0)}^{(1,1)} y\ dx + (x+2y)\ dy = \int_0^1 [x^2 + 2x(x+2x^2)]\ dx = \frac{1}{3}x^3 + \frac{2}{3}x^3 + x^4 \Big|_0^1 = 2.$$

c) Here, $dy = dx$. Thus,

$$\int_{(0,0)}^{(1,1)} y\ dx + (x+2y)\ dy = \int_0^1 4x\ dx = 2x^2\Big|_0^1 = 2.$$

The suspicion is that it does not matter what path we take. What property does this integrand have that makes the path irrelevant?

If $P(x,y)$ and $Q(x,y)$ are defined and continuous in a domain D, then $\int P\ dx + Q\ dy$ is *independent of path* if, for every pair of endpoints A and B in D, $\int_A^B P\ dx + Q\ dy$ is the same for all paths from A to B.

Theorem 46.1. *The integral $\int P\ dx + Q\ dy$ is independent of path in D iff there is a function $F \in D$ such that $\partial F/\partial x = P$ and $\partial F/\partial y = Q$.*

Proof. Suppose the intergal is independent of path in D. Then for some fixed point (x_0, y_0), define $F(x,y)$ as

$$F(x,y) = \int_{(x_0,y_0)}^{(x,y)} P\ dx + Q\ dy,$$

where the path is arbitrary from (x_0, y_0) to (x, y). Since the integral is independent of path, the function F depends only on (x, y).

For a particular (x, y) in D, choose (x_1, y) so that $x \neq x_1$ and so that the line segment from x_1 to x is in D. Then, because of independence of path,

$$F(x, y) = \int_{(x_0,y_0)}^{(x_1,y)} (P\ dx + Q\ dy) + \int_{(x_1,y)}^{(x,y)} (P\ dx + Q\ dy).$$

Thus, y is restricted to a constant value, and $F(x, y)$ is now considered a function of x. Thus, the first integral is not dependent on x, and has a constant value k; while the second can be integrated along the line segment. Hence,

$$F(x, y) = k + \int_{x_1}^{x} P(x, y)\ dx$$

or, with dummy x replaced with t,

$$F(x, y) = k + \int_{x_1}^{x} P(t, y)\ dt.$$

By the Fundamental Theorem, $\partial F/\partial x = P(x, y)$. Similarly, $\partial F/\partial y = Q$.

Now, assume there is a function $F \in D$ such that $\partial F/\partial x = P$ and $\partial F/\partial y = Q$. Then, in terms of parameter t,

$$\begin{aligned} \int_{(x_1,y_1)}^{(x_2,y_2)} P\ dx + Q\ dy &= \int_{t_1}^{t_2} \left(\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \right)\ dt \\ &= \int_{t_1}^{t_2} \frac{dF}{dt}\ dt = F|_{t_1}^{t_2} \\ &= F(x_2, y_2) - F(x_2, y_1) \end{aligned}$$

or,

$$\int_A^B P\ dx + Q\ dy = \int_A^B dF = F(B) - F(A)$$

□

Implications of Theorem 46.1.

I) This gives us an "antiderivative" of a multivariable integral. Further, the theorem implies that $\int P\ dx + Q\ dy$ *is independent of path iff* $P\ dx + Q\ dy$ *is an exact differential.*

II) F is constant in D iff $F(B) = F(A)$ for every two points $A, B \in D$.

III) In applications, such as thermodynamics, partial derivatives $P = \partial F/\partial x$ and $Q = \partial F/\partial y$ of functions representing energy and entropy are the measured functions, rather than the functions themselves. Then the function $F(x, y)$ in the proof of the theorem can be used to determine values of F at any point.

IV) The function F must also be defined and continuous in a domain containing the path; a function such as $\arctan(y/x)$ is many-valued, and so is useless for the theorem.

Example 46.2. *Evaluate* $\int_{(1,1)}^{(3,7)} y\ dx + x\ dy$.

We use the fact that $y\ dx + x\ dy$ is an exact differential to write

$$\int_{(1,1)}^{(3,7)} y\ dx + x\ dy = \int_{(1,1)}^{(3,7)} d(xy) = xy\Big|_{(1,1)}^{(3,7)} = 21 - 1 = 20.$$

Theorem 46.2. $\int P\ dx + Q\ dy$ *is independent of path in* D *iff* $\oint_C P\ dx + Q\ dy = 0$ *on every simple closed path in* D.

Since $P\ dx + Q\ dy$ can be interpreted as a vector $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then $\mathbf{F}$ defines a vector field. If $\oint_C \mathbf{F}\ d\mathbf{r} = \oint_C P\ dx + Q\ dy = 0$ then $\mathbf{F}$ determines a *conservative vector field*; so-called since all work/energy that is used equals the gain in work/energy.

Example 46.3. *Evaluate* $\oint_C y^2 \cos xy\ dx + (\sin xy + xy \cos xy)\ dy$ *on the ellipse* $9x^2 + y^2 = 9$.

Since $P\ dx + Q\ dy = dF$ where $F = y \sin xy$, we have that the integral is zero by the theorem above.

Example 46.4. *Evaluate the integral in the previous example on any path* C *from* $(0, 0)$ *to* $(\frac{3}{2}, 3\pi)$.

Since $P\ dx + Q\ dy$ is exact, we have

$$\int_{(0,0)}^{(3/2,3\pi)} d(y \sin xy) = y \sin xy\Big|_{(0,0)}^{(3/2,3\pi)} = 3\pi.$$

HOMEWORK FOR DAY 46. Page 300 #1

HOMEWORK ANSWERS. $\#1$

a) $F = x^2y$, so the integral is $x^2y\Big|_{(0,0)}^{(1,1)} = 1$.

b) $F = e^{xy}$, so the integral is $e^{xy}\Big|_{(0,0)}^{(\pi,0)} = 1$.

c) $F = \dfrac{-1}{\sqrt{x^2+y^2}}$, so the integral is

$$\left.\frac{-1}{\sqrt{x^2+y^2}}\right|_{(1,0)}^{(e^{2\pi},0)} = \frac{-1}{e^{2\pi}} + 1 = 1 - e^{-2\pi}.$$

47 Sections 5.6B and 5.7, Simply and Multiply Connected Domains

Objective. *Students will determine whether a given line integral is independent of path. Students will prove that the value of a path independent line integral on a multiply connected domain is either zero or has the same value for every path.*

Determination of path independence requires simply connected domains. A *simply connected domain* is a domain with no "holes."

Theorem 47.1 (Test for Independence of Path)**.** *Let D be a simply connected domain and let $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives in D. $\int P\ dx + Q\ dy$ is independent of path iff $\partial P/\partial y = \partial Q/\partial x$.*

Proof. Assume $\int P\ dx + Q\ dy$. Then by Theorem 46.1, $P = \partial F/\partial x$ and $Q = \partial F/\partial y$. Since P and Q have continuous derivatives in D,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Now assume $\partial P/\partial y = \partial Q/\partial x$. We choose any simple closed curve C and apply Green's Theorem:

$$\oint_C P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy = 0.$$

Green's Theorem is applicable since D is simply connected so that R is in D and both $\partial P/\partial y$ and $\partial Q/\partial x$ are continuous in R. Since C was any closed simple curve, we have that the line integral is zero for every such curve; thus, from Theorem 46.2, $\int P\ dx + Q\ dy$ is independent of path in D. □

Example 47.1. *Evaluate $\int_C (3x^2 + y)\ dx + x\ dy$ on $y = \tan x$ from $(0, 0)$ to $(\pi/4, 1)$.*

Since $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$, the integral is independent of path. Thus there is some F such that $dF = P\ dx + Q\ dy$. This F is $F = x^3 + xy$; hence,

$$x^3 + xy\Big|_{(0,0)}^{(\pi/4,1)} = \frac{\pi^3}{64} + \frac{\pi}{4} = \frac{\pi^3 + 16\pi}{64}$$

To find this function F, we use a vector interpretation:

Theorem 47.2 (Restatement of Theorem 47.1). *$\int P\ dx + Q\ dy$ is independent of path iff $P\mathbf{i} + Q\mathbf{j} = \nabla F$. Thus,*

$$\int_A^B P\ dx + Q\ dy = \int_A^B \nabla F \cdot d\mathbf{r} = F(B) - F(A)$$

Moreover, if $\mathbf{u} = P\mathbf{i} + Q\mathbf{j}$, we have $\mathbf{u} = \nabla F$ iff $\operatorname{curl} \mathbf{u} = \mathbf{0}$.

Example 47.2. *Evaluate $\int 2xy^3\ dx + 3x^2y^2\ dy$ from $(1,2)$ to $(3,-2)$.*

Since $\frac{\partial}{\partial y}(2xy^3) = \frac{\partial}{\partial x}(3x^2y^2)$, this integral is independent of path. Hence, we can integrate along any path, say the broken line with from $(0,0)$ to $(x,0)$ to (x,y). Along the first segment, $dy = y = 0$ and the integral is 0; along the second, $dx = 0$ and x is constant, giving $\int_0^y 3x^2y^2\ dy = x^2y^3 = F(x,y)$. Hence,

$$\int_{(1,2)}^{(3,-2)} 2xy^3\ dx + 3x^2y^2\ dy = x^2y^3\Big|_{(1,2)}^{(3,-2)} = -80.$$

Example 47.3. *See Example 2 on page 295.*

Example 47.4. *Evaluate $\int_{(1,0)}^{(-1,0)} (2xy-1)\ dx + (x^2+6y)\ dy$ on the semi-circle $y = \sqrt{1-x^2}$.*

Since $\partial P/\partial y = \partial Q/\partial x$, the integral is independent of path; $F = x^2y + 3y^2 - x$ and we calculate $F\Big|_{(1,0)}^{(-1,0)} = 2$.

Example 47.5. *Evaluate $\oint [\sin xy + xy\cos xy]\ dx + x^2 \cos xy\ dy$ on the circle $x^2 + y^2 = 1$.*

Since $\partial P/\partial y = \partial Q/\partial x$, the integral is independent of path and so is zero.

Theorem 47.3. *Let P and Q have continuous partials in a multiply connected domain D. If R is a closed region in D whose boundary consists of finitely many simple closed curves C_i, then*

$$\sum_{i=1}^{n} \oint_{C_i} P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dx\ dy.$$

Moreover, this double integral is zero if $\partial P/\partial y = \partial Q/\partial x$.

As a result of this theorem, we have that *either $\oint_C P\ dx + Q\ dy$ is zero when C does not include a "hole" and is a nonzero constant value otherwise.*

Example 47.6. *Evaluate* $\oint \frac{y^3\ dx - xy^2\ dy}{(x^2+y^2)^2}$ *around the ellipse* $2x^2+3y^2=1$.

Since we have $\partial P/\partial y = \partial Q/\partial x$ except at the origin, we may use any closed curve to evaluate the integral; we choose $x^2+y^2=1$ and then use the parametrization $x=\cos t,\ y=\sin t$. Hence,

$$\begin{aligned}\oint y^3\ dx - xy^2\ dy &= \int_0^{2\pi} (-\sin^4 t - \cos^2 t \sin^2 t)\ dt \\ &= -\int_0^{2\pi} \sin^2 t\ dt = -\pi\end{aligned}$$

Hence, the integral has the the value $-\pi$ around the ellipse as well.

HOMEWORK FOR DAY 47. Page 300 #2 parts b and d, Page 301 #3 parts b and d, #4, #6 part a, #7

HOMEWORK ANSWERS. #2 b) This integral is independent of path; $F = x^3/y$ so $F\Big|_{(0,2)}^{(1,3)} = \frac{1}{3}$.

#2 d) This integral is independent of path; $F = \tan x \tan y$ so $F\Big|_{(0,0)}^{(\pi/4,\pi/4)} = 1$.

#3 b) This integral is independent of path; $F = -\arctan\left(\frac{y}{x-1}\right)$ so the integral is -2π.

#3 d) We write $\oint xy^6\ dx + 3x^2y^5\ dy + \oint 6x\ dy$ so that, since the first integral is independent of path and so is zero, we only need consider the second integral, which is $6A$ where A is the area of the ellipse. Hence, the integral is 12π.

#4 From $(1,0)$ to $(2,2)$, we move through an angle of $\pi/4$; so all answers are of the form $\pi/4 + 2\pi k$.

#6 a) This integral is independent of path; $F = x^2y - \frac{1}{3}y^3$ so $F\Big|_{(1,1)}^{(x,y)} = x^2y - \frac{1}{3}y^3 - 1 + \frac{1}{3} = x^2y - \frac{1}{3}(y^3 + 2)$.

#7 Since the integral is independent of path, we use $x^2 + y^2 = 1$ as C. Then with the parametrization $x = \cos t,\ y = \sin t$ we have

$$\int_0^{2\pi} (-\cos^2 t \sin^2 t - \cos^4 t)\ dt = -\int_0^{2\pi} \cos^2 t = -\pi.$$

48 Sections 5.8 and 5.9, Line Integrals, Surfaces, and Orientability in Space

Objective. *Students will evaluate line integrals in space. Students will use normal and tangent vectors to describe surfaces in space and their orientability.*

Basic definitions of line integrals still apply to those in space.

$\int_C X\ dx + Y\ dy + Z\ dz$ is a line integral in space. The vector interpretation is $\int_C \mathbf{u}\cdot d\mathbf{r} = \int_C u_T\ ds$ where $\mathbf{u} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ and $\mathbf{T}$ is the unit tangent vector.

Example 48.1. *Evaluate* $\int_{(1,0,0)}^{(1,0,2\pi)} z\ dx + y\ dy + x\ dz$ *where* C *is the curve defined by* $x = \cos t,\ y = \sin t,\ z = t$ *for* $0 \le t \le 2\pi$.

We use the parametrization of C to find the integral:

$$\begin{aligned}\int_{(1,0,0)}^{(1,0,2\pi)} z\ dx + y\ dy + x\ dz &= \int_0^{2\pi} (-t\sin t + \sin t\cos t + \cos t)\ dt \\ &= t\cos t - \sin t - \frac{1}{4}\cos 2t + \sin t\Big|_0^{2\pi} = 2\pi\end{aligned}$$

Example 48.2. *Evaluate* $\int_C yz\ dx + xz\ dy + xy\ dz$ *where* C *is the "twisted cubic" defined by* $x = t,\ y = t^2,\ z = t^3$ *for* $0 \le t \le 2$.

We have $dx = dt,\ dy = 2t\ dt,\ dz = 3t^2\ dt$. Thus,

$$\begin{aligned}\int_C yz\ dx + xz\ dy + xy\ dz &= \int_0^2 t^5\ dt + 2t^5\ dt + 3t^5\ dt \\ &= \int_0^2 6t^5\ dt = t^6\big|_0^2 = 64.\end{aligned}$$

Surfaces can be described by any of

$$z = f(x,y), \qquad F(x,y,z) = 0, \qquad x = f(u,v),\ y = g(u,v),\ z = h(u,v)$$

For a surface $z = f(x,y)$, we have the area of S as

$$\iint\limits_{R_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\ dy.$$

As we have the arc length element ds, we have the surface area element $d\sigma$.

For a surface defined parametrically, we have $\iint_{R_{uv}} \sqrt{EG - F^2}\, du\, dv$.

We assume surfaces are piecewise smooth. We also define a "direction"—we choose normal $\mathbf{n}$ at each point so $\mathbf{n}$ varies continuously on S (one can then define a positive direction for angle measure). —[[LARSON 114, 116]]— Choose $\mathbf{T}$ in the direction chosen and form an *inner normal* $\mathbf{N}$ so $\mathbf{N}$ always points to the left of the boundary C of the surface. Then we have an *orientation*. Not all surfaces have an orientation; i.e., the Möbius strip is *one-sided* and so is not orientable. —[[THOMAS 13.46]]—

HOMEWORK FOR DAY 48. Page 312 #1 parts a, b, d, and e

HOMEWORK ANSWERS. #1

a) The integral becomes

$$\int_0^{2\pi} (-t\sin t + \cos^2 t + \sin t)\, dt = t\cos t - \sin t + \frac{1}{2}t + \frac{1}{4}\sin 2t - \cos t\Big|_0^{2\pi}$$
$$= 3\pi$$

b) Since the path is the straight line, we find $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$ as the vector along the line; thus the line has parametrization $x = z = 1 + t,\ y = 3t$ for $0 \le t \le 1$. Thus, the integral is

$$\int_0^1 \left[(1+t)^2 - 3(1+t)^2 + 9t^2\right]\, dt = \int_0^1 (-2t^2 - 4t - 2 + 9t^2)\, dt$$
$$= \frac{7}{3}t^3 - 2t^2 - 2t\Big|_0^1 = -\frac{5}{3}$$

d) The integral becomes

$$\int_0^{2\pi} (-4\sin^3 t\cos t + 4\cos^3 t\sin t)\, dt = -\sin^4 t - \cos^4 t\big|_0^{2\pi} = 0$$

e) We compute $\operatorname{curl}\mathbf{v} = -2z\mathbf{i} - 2x\mathbf{j} - 2y\mathbf{k}$ and the integral becomes

$$\int_0^1 \left(-4(1+t^3) - 4t(2t+1) - 2t^2(3t^2)\right)\, dt$$
$$= \int_0^1 (-6t^4 - 4t^3 - 8t^2 - 4t - 4)\, dt$$
$$= -\frac{6}{5}t^5 - t^4 - \frac{8}{3}t^3 - 2t^2 - 4t\Big|_0^1 = -\frac{163}{15}$$

49 Section 5.10, Surface Integrals

Objective. *Students will evaluate surface integrals of functions given in standard and vector forms.*

Let S be smooth surface and $H(x, y, z)$ be defined and continuous on S. Then the surface integral is analogous to the line integral; i.e.,

$$\iint\limits_S H\ d\sigma = \lim_{n\to\infty} \sum_{i=1}^{n} H\Delta_i\sigma$$

where $\Delta_i\sigma$ is the area of the ith piece. Note that $d\sigma$ is a surface area element, similar to ds for the arc length element. Hence,

$$d\sigma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\ dy$$

Thus if S is of the form $z = f(x, y)$ we have the double integral

$$\iint\limits_S H\ d\sigma = \iint\limits_{R_{xy}} H(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dx\ dy$$

Example 49.1. *Evaluate* $\iint_S x^2 z\ d\sigma$ *where* S *is the portion of the cone* $z^2 = x^2 + y^2$ *that lies between the planes* $z = 1$ *and* $z = 4$.

We first compute the partials. We have

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

Thus,

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

and the integral becomes

$$\iint_S x^2 z\ d\sigma = \sqrt{2} \iint_{R_{xy}} x^2\sqrt{x^2 + y^2}\ dx\ dy.$$

Now we use polar to integrate. We note that $0 \le z \le 4$ implies $0 \le r \le 4$, and that is the Jacobian is r.

$$\begin{aligned}
&= \sqrt{2} \int_0^{2\pi} \int_1^4 r(r^2 \cos^2 \theta) r \; dr \; d\theta \\
&= \sqrt{2} \int_0^{2\pi} \frac{1}{5} r^5 \cos^2 \theta \Big|_1^4 \; d\theta \\
&= \frac{1023\sqrt{2}}{5} \int_0^{2\pi} \cos^2 \theta \; d\theta \\
&= \frac{1023\sqrt{2}}{5} (\pi) \approx 909
\end{aligned}$$

If S is parametric, then we have

$$\iint_S H d\sigma = \iint_{R_{uv}} H\big(f(u,v), g(u,v), h(u,v)\big)\sqrt{EG - F^2} \; du \; dv$$

In either case, we have a form similar to $\int X \; dx + Y \; dy + Z \; dz$ for line integrals: with continuous unit normal $\mathbf{n} = \cos\alpha \mathbf{i} + \cos\beta \mathbf{j} + \cos\gamma \mathbf{k}$ and continuous vector function $\mathbf{v} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ defined on S, we have

$$\begin{aligned}
\iint_S \mathbf{v} \cdot \mathbf{n} \; d\sigma &= \iint_S (L \cos\alpha + M \cos\beta + N \cos\gamma) \; d\sigma \\
&= \iint_S L \; dy \; dz + M \; dz \; dx + N \; dx \; dy
\end{aligned}$$

as the integral over the surface.

Theorem 49.1. *The evaluation of a surface integral is given by the following.*

I) If S is given in the form $z = f(x,y)$ with normal vector $\mathbf{n}$, then

$$\iint_S L \; dy \; dz + M \; dz \; dx + N \; dx \; dy = \pm \iint_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N \right) \; dx \; dy$$

where the sign is used when $\mathbf{n}$ is the upper/lower normal.

II) If S is given by $x = f(u,v),\ y = g(u,v),\ z = h(u,v)$ with normal vector $\mathbf{n}$, then

$$\iint_S L\ dy\ dz + M\ dz\ dx + N\ dx\ dy$$
$$= \pm \iint_{R_{uv}} \left[L\frac{\partial(y,z)}{\partial(u,v)} + M\frac{\partial(z,x)}{\partial(u,v)} + N\frac{\partial(x,y)}{\partial(u,v)} \right]\ du\ dv$$

where the sign of $\mathbf{n}$ is given by $\mathbf{n} = \dfrac{\mathbf{P}_u \times \mathbf{P}_v}{|\mathbf{P}_u \times \mathbf{P}_v|}$ where $\mathbf{P}_u = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k}$ and $\mathbf{P}_v = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k}$.

Proof. We prove the first case. For a surface $z = f(x,y)$, we have the tangent plane is $z - z_1 = \frac{\partial z}{\partial x}(x - x_1) + \frac{\partial z}{\partial y}(y - y_1)$ and hence that the unit normal is

$$\mathbf{n} = \pm \frac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}},$$

with the sign according to upper or lower normal. Therefore,

$$\iint_S L\ dy\ dz + M\ dz\ dx + N\ dx\ dy$$
$$= \iint_S \left[(L\mathbf{i} + M\mathbf{j} + N\mathbf{k}) \cdot \mathbf{n} \right]\ d\sigma$$
$$= \pm \iint_S \frac{-L\frac{\partial z}{\partial x}\mathbf{i} - M\frac{\partial z}{\partial y}\mathbf{j} + N\mathbf{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}\ d\sigma$$
$$= \pm \iint_{R_{xy}} \left(-L\frac{\partial z}{\partial x} - M\frac{\partial z}{\partial y} + N \right)\ dx\ dy$$

□

Example 49.2. *Evaluate $\iint_S x\ dy\ dz + y\ dz\ dx + z\ dx\ dy$ for S as the hemisphere $z = \sqrt{1 - x^2 - y^2},\ x^2 + y^2 \leq 1$ and $\mathbf{n}$ is the upper normal.*

$$
\begin{aligned}
&\iint\limits_S x\ dy\ dz + y\ dz\ dx + z\ dx\ dy \\
&\quad = \iint\limits_R \left[-x\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right) - y\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right) + \sqrt{1-x^2-y^2}\right]\ dx\ dy \\
&\quad = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}}\ dx\ dy \\
&\quad = \int_{-1}^{1} \arcsin \frac{y}{\sqrt{1-x^2}}\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \pi\ dx = 2\pi
\end{aligned}
$$

Example 49.3. *Evaluate the above parametrically using* $x = \sin u \cos v,\ y = \sin u \sin v,\ z = \cos u.$

First, we form the Jacobians:

$$
\begin{aligned}
\begin{vmatrix} \cos u \sin v & \sin u \cos v \\ -\sin u & 0 \end{vmatrix} &= \sin^2 u \cos v \\
\begin{vmatrix} -\sin u & 0 \\ \cos u \cos v & -\sin u \sin v \end{vmatrix} &= \sin^2 u \sin v \\
\begin{vmatrix} \cos u \cos v & -\sin u \sin v \\ \cos u \sin v & \sin u \cos v \end{vmatrix} &= \cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v \\
&= \cos u \sin u
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\iint\limits_S x\ dy\ dz + y\ dz\ dx + z\ dx\ dy \\
&\quad = \iint\limits_R \big[\sin u \cos v(\sin^2 u \cos v) + \sin u \sin v(\sin^2 u \sin v) \\
&\qquad\qquad + \cos u(\cos u \sin u)\big]\ du\ dv \\
&\quad = \iint\limits_R \left[\sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u\right]\ du\ dv \\
&\quad = \int_0^{\pi}\int_0^{\pi} \sin u\ du\ dv = \int_0^{\pi} -\cos u\big|_0^{\pi}\ dv = \int_0^{\pi} 2\ dv = 2\pi
\end{aligned}
$$

The surface integral can also be defined similarly to the vector line integral $\int \mathbf{u} \cdot d\mathbf{r}$. We use a differential area vector:

$$d\boldsymbol{\sigma} = \mathbf{n}\ d\sigma = \cos\alpha\ d\sigma\mathbf{i} + \cos\beta\ d\sigma\mathbf{j} + \cos\gamma\ d\sigma\mathbf{k}$$

Thus,

$$\iint\limits_S L\ dy\ dz + M\ dz\ dx + N\ dx\ dy = \iint\limits_S \mathbf{v} \cdot \mathbf{n}\ d\sigma = \iint\limits_S \mathbf{v} \cdot d\boldsymbol{\sigma}.$$

Example 49.4. *Evaluate* $\iint_S \mathbf{w} \cdot \mathbf{n}\ d\sigma$ *if* $\mathbf{w} = 2yz\mathbf{i} + 2xz\mathbf{j} + xy\mathbf{k}$ *on the surface* $z = 1 - x^2 - y^2,\ x^2 + y^2 \le 1$, *and* $\mathbf{n}$ *is the upper normal.*

$$\begin{aligned}
&\iint\limits_S \mathbf{w} \cdot \mathbf{n}\ d\sigma \\
&\quad = \iint\limits_S 2yz\ dy\ dz + 2xz\ dz\ dx + xy^2\ dx\ dy \\
&\quad = \iint\limits_S \left(-2yz(-2x) - 2xz(-2y) + xy^2\right)\ dx\ dy \\
&\quad = \iint\limits_S \left(4xy(1 - x^2 - y^2) + 4xy(1 - x^2 - y^2) + xy^2\right)\ dx\ dy \\
&\quad = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(8xy - 8x^3y - 8xy^3 + xy^2\right)\ dy\ dx \\
&\quad = \int_{-1}^{1} \left(4xy^2 - 4x^3y^2 - 2xy^4 + \frac{xy^3}{3}\right)\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dx \\
&\quad = \int_{-1}^{1} \frac{2}{3}x(1 - x^2)^{3/2}\ dx = 0
\end{aligned}$$

HOMEWORK FOR DAY 49. Page 313 #5 parts a, b, and d, #6 part b, #7 part a

HOMEWORK ANSWERS. #5 a) A normal vector to the surface is $\mathbf{n} = \mathbf{i}+\mathbf{j}+\mathbf{k}$. The plane is then $x-1+y+z=0$ or $z = 1-x-y$, for $x, y \geq 0$ and $x+y \leq 1$. Hence,

$$\begin{aligned}
&\iint_S x\ dy\ dz + y\ dz\ dx + z\ dx\ dy \\
&\qquad = \iint_S (-x(-1) - y(-1) + 1 - x - y)\ \ dx\ dy \\
&\qquad = \int_0^1 \int_0^{1-x}\ dy\ dx \\
&\qquad = \int_0^1 (1-x)\ dx = \frac{1}{2}
\end{aligned}$$

#5 b)

$$\begin{aligned}
&\iint_S dy\ dz + dz\ dx + dx\ dy \\
&\qquad = \iint_S \left(\frac{x}{\sqrt{1-x^2-y^2}} + \frac{y}{\sqrt{1-x^2-y^2}} + 1 \right)\ dx\ dy \\
&\qquad = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\frac{x/\sqrt{1-x^2}}{\sqrt{1-y^2/(1-x^2)}} + \frac{y}{\sqrt{1-x^2-y^2}} + 1 \right)\ dx\ dy \\
&\qquad = \int_{-1}^1 \left(x \arcsin \frac{y}{\sqrt{1-x^2}} - \sqrt{1-x^2-y^2} + y \right) \Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dx \\
&\qquad = \int_{-1}^1 \left(x\pi + 2\sqrt{1-x^2} \right)\ dx \\
&\qquad = \pi
\end{aligned}$$

#6 b) The Jacobians are the same as from the example above, so the integral is

$$\iint\limits_S \left[\sin^2 u \cos v + \sin^2 u \sin v + \cos u \sin u\right]\ du\ dv$$

$$= \int_0^\pi \int_0^\pi \left[\sin^2 u(\cos v + \sin v) + \cos u \sin u\right]\ du\ dv$$

$$= \int_0^\pi \int_0^\pi \left[\left(\frac{1}{2} - \frac{1}{2}\cos 2u\right)(\cos v + \sin v) + \frac{1}{2}\sin 2u\right]\ du\ dv$$

$$= \int_0^\pi \left[\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)(\cos v + \sin v) - \frac{1}{4}\cos 2u\right]\Bigg|_0^\pi\ dv$$

$$= \int_0^\pi \frac{\pi}{2}(\cos v + \sin v)\ dv$$

$$= \pi$$

#7 a)

$$\iint\limits_S xy^2z\ dy\ dz - 2x^3\ dz\ dx + yz^2\ dx\ dy$$

$$= \iint\limits_S \left(-xy^2z(-2x) + 2x^3(-2y) + yz^2\right)\ dx\ dy$$

$$= \iint\limits_S \left(2x^2y^2(1-x^2-y^2) - 4x^3y + y(1-x^2-y^2)^2\right)\ dx\ dy$$

$$= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x^2y^2 - 2x^4y^2 - 2x^2y^4 + y(1-x^2-y^2)^2)\ dy\ dx$$

$$= \int_{-1}^1 \left(\frac{2}{3}x^2y^3 - \frac{2}{3}x^4y^3 - \frac{2}{5}x^2y^5 - 2x^3y^2 - \frac{1}{6}(1-x^2-y^2)^3\right)\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\ dx$$

$$= \int_{-1}^1 \left(\frac{4}{3}(1-x^2)^{3/2}(x^2-x^4) - \frac{4}{5}x^2(1-x^2)^{5/2}\right)\ dx$$

$$= \int_{-1}^1 \frac{8}{15}x^2(1-x^2)^{5/2}\ dx = \frac{\pi}{48}$$

50 Section 5.11, Gauss' Divergence Theorem

Objective. *Students will evaluate surface integrals by applying Gauss' Divergence Theorem.*

Gauss' Divergence Theorem allows reduction of a surface integral to a triple integral.

Theorem 50.1 (Gauss' Divergence Theorem). *Let* $\mathbf{v} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ *be a vector field in a domain* D *of space; let* L, M, N *be continuous and have continuous derivatives in* D. *Let* $S \subseteq D$ *be a piecewise smooth surface that forms the complete boundary of a bounded closed region* $R \subseteq D$. *Let* $\mathbf{n}$ *be the outer normal of* S. *Then*

$$\iint_S \mathbf{v_n}\, d\boldsymbol{\sigma} = \iiint_R \operatorname{div}\mathbf{v}\; dx\; dy\; dz;$$

that is,

$$\iint_S L\; dy\; dz + M\; dz\; dx + N\; dx\; dy = \iiint_R \left(\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right)\; dx\; dy\; dz.$$

Proof. We prove $\iint_S N\; dx\; dy = \iiint_R \frac{\partial N}{\partial z}\; dx\; dy\; dz$ since the other two relations are done exactly the same.

Assume R is representable in the form $f_1(x,y) \le z \le f_2(x,y)$ for $(x,y) \in R_{xy}$ where R_{xy} is a bounded closed region in the xy-plane bounded by a simple closed curve C. The surface S is then composed of three parts:

$$S_1 : z = f_1(x,y) \quad S_2 : z = f_2(x,y) \quad S_3 : f_1(x,y) \le z \le f_2(x,y).$$

(S_1 is the bottom, S_2 is the top, S_3 forms the sides.) —[[Larson 118]]— Let γ be the angle between $\mathbf{n}$ and $\mathbf{k}$. Then

$$\iint_S N\; dx\; dy = \iint_S N\cos\gamma\; d\sigma.$$

Note that $\gamma = \pi/2$ so that along S_3, $\cos\gamma = 0$. Let γ' be the angle between the upper normal and $\mathbf{k}$. Then along S_2, $\gamma = \gamma'$—so that $\cos\gamma = \cos\gamma'$—and along S_1, $\gamma = \pi - \gamma'$—so that $\cos\gamma = \cos(\pi - \gamma') = -\cos\gamma'$. Since

$d\sigma = \sec\gamma'\ dx\ dy$ on S_1 and S_2, we have

$$\iint\limits_S N\ dx\ dy = \iint\limits_{S_1} N\ dx\ dy + \iint\limits_{S_2} N\ dx\ dy + \iint\limits_{S_3} N\ dx\ dy$$
$$= -\iint\limits_{R_{xy}} N\cos\gamma'\sec\gamma'\ dx\ dy + \iint_{R_{xy}} N\cos\gamma'\sec\gamma'\ dx\ dy$$
$$= \iint\limits_{R_{xy}} \big[N(x,y,f_2(x,y)) - N(x,y,f_1(x,y))\big]\ dx\ dy$$

Now, the triple integral can be evaluated as

$$\iiint\limits_R \frac{\partial N}{\partial z}\ dx\ dy\ dz = \iint\limits_{R_{xy}} \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial N}{\partial z}\ dz\ dx\ dy$$
$$= \iint\limits_{R_{xy}} \big[N(x,y,f_2(x,y)) - N(x,y,f_1(x,y))\big]\ dx\ dy$$

Hence, the result follows. □

Example 50.1. *Evaluate* $\iint_S xy\ dy\ dz + (y^2 + e^{xz^2})\ dz\ dx + \sin(xy)\ dx\ dy$ *where* S *is the surface of the region bounded by* $z = 1 - x^2$, $z = y = 0$, *and* $y + z = 2$.

By the Divergence Theorem, we have

$$\iiint\limits_R (y + 2y + 0)\ dx\ dy\ dz = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\ dy\ dz\ dx$$
$$= 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2-z)^2}{2}\ dz\ dx$$
$$= 3\int_{-1}^{1} -\frac{(2-z)^3}{6}\Bigg|_0^{1-x^2}\ dx$$
$$= -\frac{1}{2}\int_{-1}^{1}[(x^2+1)^3 - 8]\ dx$$
$$= -\frac{1}{2}\int_{-1}^{1}(x^6 + 3x^4 + 3x^2 - 7)\ dx = \frac{184}{35}$$

Example 50.2. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\ d\sigma$ *if* $\mathbf{F} = (x^3+\sin z)\mathbf{i} + (x^2y+\cos z)\mathbf{j} + e^{x^2+y^2}\mathbf{k}$ *and* S *is the surface of the region bounded by* $z = 4-x^2$, $y+z=5$, *and* $z=y=0$.

By the Divergence Theorem, we have

$$\begin{aligned}\iiint\limits_R (3x^2+x^2+0)\ dx\ dy\ dz &= \iiint\limits_R 4x^2\ dx\ dy\ dz \\ &= \int_{-2}^{2}\int_0^{4-x^2}\int_0^{5-z} 4x^2\ dy\ dz\ dx \\ &= \int_{-2}^{2}\int_0^{4-x^2} 4x^2(5-z)\ dz\ dx \\ &= \int_{-2}^{2} (20x^2z-2x^2z^2)\Big|_0^{4-x^2}\ dx \\ &= \int_{-2}^{2}(48x^2-4x^4-2x^6)\ dx = \tfrac{4608}{35}\approx 131.7\end{aligned}$$

Example 50.3. *Evaluate* $\iint_S \nabla F\cdot\mathbf{n}\ d\sigma$ *if* $F = 2x^2-y^2-z^2$ *with* S *as the unit sphere.*

We have

$$\iint\limits_S 4x\ dy\ dz - 2y\ dx\ dz - 2z\ dx\ dy = \iiint_R (4-2-2)\ dx\ dy\ dz = 0.$$

Example 50.4. *Evaluate* $\iint_S \mathbf{v_n}\ d\sigma$ *where* $\mathbf{v} = 3x\mathbf{i} + y^2\mathbf{j} + 2z^3\mathbf{k}$ *and* S *is the surface of the unit cube in the first octant.*

By the Divergence Theorem, we have

$$\begin{aligned}\iiint\limits_R (3+2y+6z^2)\ dx\ dy\ dz &= \int_0^1\int_0^1\int_0^1 (3+2y+6z^2)\ dx\ dy\ dz \\ &= \int_0^1\int_0^1 (3+2y+6z^2)\ dy\ dz \\ &= \int_0^1 \left[(3+6z^2)y+y^2\right]\Big|_0^1 dz \\ &= \int_0^1 (4+6z^2)\ dz = 4z+2z^3\Big|_0^1 = 6\end{aligned}$$

HOMEWORK FOR DAY 50. Page 319 #1 parts a, b, c, and d

HOMEWORK ANSWERS. #1 a) Using the Divergence Theorem,

$$\iiint\limits_R 3\ dz\ dy\ dx = 3 \iiint\limits_R dz\ dy\ dx = 3\left(\frac{4\pi}{3}\right) = 4\pi$$

#1 c) We have $\iiint\limits_R 0\ dz\ dy\ dx = 0.$

#1 d) We have

$$\iint\limits_S 2x\ dy\ dz + 2y\ dx\ dz + 2z\ dx\ dy = \iiint\limits_R 6\ dx\ dy\ dz = 6V$$

where V is the volume of S.

51 Section 5.12, Stokes' Theorem

Objective. *Students will evaluate line integrals in space by applying Stokes' Theorem.*

Recall Green's Theorem can be written as $\oint_C u_T \, ds = \iint_R \operatorname{curl}_z \mathbf{u} \, dx \, dy$. This suggests we can write $\int_C u_T \, ds = \iint_S \operatorname{curl}_n \mathbf{u} \, d\sigma$ where $\mathbf{n}$ is the normal to S, the planar surface bounded by C.

The surface integral $\int_S \operatorname{curl}_n \mathbf{u} \, d\sigma$ has the same value for all surfaces with boundary C. If S_1 and S_2 have the same boundary C with no other common points, we compute:

$$\begin{aligned}\iint\limits_{S_1} \operatorname{curl}_n \mathbf{u} \, d\sigma - \iint\limits_{S_2} \operatorname{curl}_n \mathbf{u} \, d\sigma &= \iint\limits_{S} \operatorname{curl}_n \mathbf{u} \, d\sigma \\ &= \pm \iiint\limits_{R} \operatorname{div} \operatorname{curl} \mathbf{u} \, dx \, dy \, dz = 0\end{aligned}$$

Theorem 51.1 (Stokes' Theorem)**.** *Let S in domain D be a piecewise smooth orientable surface with boundary C a piecewise smooth simple closed curve. Let $\mathbf{u} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}$ be a vector field with continuous and differentiable components in D. Then*

$$\int\limits_C u_T \, ds = \iint\limits_S \operatorname{curl} \mathbf{u} \cdot \mathbf{n} \, d\sigma$$

where $\mathbf{n}$ is the unit normal on S. In other words,

$$\begin{aligned}&\int\limits_C L \, dx + M \, dy + N \, dz \\ &\quad = \iint\limits_S \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}\right) dy \, dz + \left(\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}\right) dz \, dx + \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx \, dy\end{aligned}$$

Proof. —[[Larson 120]]— We prove $\int_C L \, dx = \iint_S \frac{\partial L}{\partial z} \, dz \, dx - \frac{\partial L}{\partial y} \, dx \, dy$, where S is of the form $z = f(x, y)$. Then C has a projection C_{xy} in the xy-plane; if we choose $\mathbf{n}$ as the upper normal, then by Green's Theorem, we have

$$\int\limits_C L(x,y,z) \, dx = \int\limits_{C_{xy}} L[x, y, f(x,y)] \, dx = -\iint\limits_{R_{xy}} \left(\frac{\partial L}{\partial y} + \frac{\partial L}{\partial z}\frac{\partial f}{\partial y}\right) dx \, dy.$$

However, we also can evaluate the following:

$$\int\limits_C \frac{\partial L}{\partial z}\,dz\,dx - \frac{\partial L}{\partial y}\,dx\,dy = \iint\limits_{R_{xy}} \left(-\frac{\partial L}{\partial z}\frac{\partial f}{\partial y} - \frac{\partial L}{\partial x}\right) dx\,dy;$$

thus, the result follows. □

We have a new interpretation of the curl: by Stokes' and the Mean Value Theorem,

$$\int\limits_{C_r} u_T\,ds = \iint\limits_S \text{curl}_n\,\mathbf{u}\,d\sigma = \text{curl}_n\,\mathbf{u}(x_0, y_0, z_0)A_r$$

for some point (x_0, y_0, z_0) where A_r is the area of the region bounded by C_r. Hence,

$$\text{curl}_n\,\mathbf{u}(x_0, y_0, z_0) = \frac{1}{A_r}\int\limits_{C_r} u_T\,ds$$

where the integral is called the *circulation* around C_r. *So the curl is the circulation per unit area.*

Example 51.1. *Evaluate* $\int_C u_T\,ds$ *where* $\mathbf{u} = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ *and* C *is the curve of intersection of* $y + z = 2$ *and* $x^2 + y^2 = 1$.

$$\begin{aligned}
\int\limits_C u_T\,ds &= \iint_S 0\,dy\,dz + 0\,dz\,dx + (1-2y)\,dx\,dy \\
&= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1-2y)\,dy\,dx \\
&= \int_{-1}^{1} (y - y^2)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} \left[2\sqrt{1-x^2} - 2(1-x^2)\right]\,dx \\
&= 2\left(\frac{\pi}{2}\right) - 2\left(x - \frac{1}{3}x^3\right)\Bigg|_{-1}^{1} \\
&= \pi
\end{aligned}$$

Example 51.2. *Evaluate* $\int_C u_T\ ds$ *where* $\mathbf{u} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ *and* S *is inside the sphere* $x^2 + y^2 + z^2 = 4$ *but above the cylinder* $x^2 + y^2 = 1$.

First, we find C: $z^2 = 3$ so $z = \sqrt{3}$. Hence, C is the circle $x^2 + y^2 = 1,\ z = \sqrt{3}$. Then

$$\int_C u_T\ ds = \iint_S (x - x)\ dy\ dx + (y - y)\ dz\ dx + (z - z)\ dx\ dy = 0.$$

HOMEWORK FOR DAY 51. Page 330 #1

HOMEWORK ANSWERS. #1 a)

$$\iint\limits_S 6\ dx\ dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 6\ dx\ dy$$
$$= \int_{-1}^{1} 6y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$$
$$= \int_{-1}^{1} 12\sqrt{1-x^2}\ dx$$
$$= 12\left(\frac{\pi}{2}\right) = 6\pi$$

#1 b)

$$\iint\limits_S (2x^2y - 2x^2y)\ dy\ dz + (2xy^2 - 2xy^2)\ dz\ dx + (4xyz - 4xyz)\ dx\ dy = 0$$

52 Section 5.13, Integrals Independent of Path

Objective. *Students will determine whether a given line integral in space is independent of path and use the property of independence of path to help evaluate line integals in space.*

Theorem 52.1 (Analogue of Theorem 46.1). *Let* $\mathbf{u} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *be a vector field with continuous components in a domain* D *of space. The line integral* $\int u_T\ ds = \int X\ dx + Y\ dy + Z\ dz$ *is independent of path in* D *if and only if there is a function* $F(x, y, z)$ *defined in* D *such that* $\frac{\partial F}{\partial x} = X,\ \frac{\partial F}{\partial y} = Y,\ \frac{\partial F}{\partial z} = Z$.

In other words, the line integral is independent of path if and only if $\mathbf{u} = \nabla F$.

The implication of the above theorem is that if $X\ dx + Y\ dy + Z\ dz$ is an exact differential of F, then the integral can be easily evaluated: $\int_A^B X\ dx + Y\ dy + Z\ dz = \int_A^B dF = F(B) - F(A)$.

Theorem 52.2 (Analogue of Theorem 46.2). *Let* X, Y, Z *be continuous in domain* D *of space. The line integral* $\int X\ dx + Y\ dy + Z\ dz$ *is independent of path in* D *if and only if* $\int_C X\ dx + Y\ dy + Z\ dz = 0$ *on every simple closed curve in* D.

Theorem 52.3 (Analogue of Theorem 47.1). *Let* $\mathbf{u} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *be a vector field with continuous partial derivatives in a simply connected domain* D *of space. Then* $\int u_T\ ds = \int X\ dx + Y\ dy + Z\ dz$ *is independent of path in* D *if and only if* $\frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z},\ \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x},\ \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}$.

In other words, the integral is independent of path if and only if $\operatorname{curl} \mathbf{u} = \mathbf{0}$.

Proof. Assume the intergal is independent of path. Then by Theorem 52.1, $\mathbf{u} = \nabla F$. Thus, $\operatorname{curl} \mathbf{u} = \operatorname{curl} \nabla F = \mathbf{0}$, by the indentity from Chapter 3.

Now assume that $\operatorname{curl} \mathbf{u} = \mathbf{0}$. Since D is simply connected, C forms the boundary of a piecewise smooth oriented surface $S \in D$. Stokes' Theorem can then be applied to get $\int u_T\ ds = \iint_S \operatorname{curl}_n \mathbf{u} d\sigma = 0$ on every simple closed curve C. □

A vector field $\mathbf{u}$ whose components have continuous derivatives with $\operatorname{curl} \mathbf{u} = \mathbf{0}$ is called *irrotational* in D. In fact, the following statements are equivalent.

$$\begin{gathered}\mathbf{u} \text{ is irrotational in } D \\ \int_C u_T \; ds = 0 \text{ for every simple closed curve in } D \\ \int u_T \; ds \text{ is independent of path in } D \\ \mathbf{u} = \nabla F \text{ in } D.\end{gathered}$$

Theorem 52.4. *Let* $\mathbf{u} = L\mathbf{i}+M\mathbf{j}+N\mathbf{k}$ *be a vector field with continuous partial derivatives in a spherical domain* D*. If* $\operatorname{div}\mathbf{u} = 0$ *in* D*, then* $\frac{\partial L}{\partial x}+\frac{\partial M}{\partial y}+\frac{\partial N}{\partial z} = 0$ *in* D *and a vector field* $\mathbf{v} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *can be found such that* $\operatorname{curl}\mathbf{v} = \mathbf{u}$ *in* D*; in other words,*

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = L, \quad \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} = M, \quad \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} = N$$

in D*.*

Proof. Let D have center at P_0; without loss of generality, assume P_0 to be the origin. If $P_1(x_1, y_1, z_1)$ is an arbitrary point of D, let

$$\begin{aligned}X(x_1,y_1,z_1) &= \int_0^1 [zM(x,y,z) - yN(x,y,z)] \; dt, \\ Y(x_1,y_1,z_1) &= \int_0^1 [xN(x,y,z) - zL(x,y,z)] \; dt, \\ Z(x_1,y_1,z_1) &= \int_0^1 [yL(x,y,z) - xM(x,y,z)] \; dt,\end{aligned}$$

where $x = x_1t,\ y = y_1t,\ z = z_1t$. As t varies from 0 to 1, the point (x, y, z) varies from P_0 to P_1 on the segment P_0P_1; so (x, y, z) remains in D. Thus, by Leibniz's Rule and the chain rule,

$$\begin{aligned}\frac{\partial Z}{\partial y_1} &= \int_0^1 \left[y\frac{\partial L}{\partial y}\frac{\partial y}{\partial y_1} + \frac{\partial y}{\partial y_1}L - x\frac{\partial M}{\partial y}\frac{\partial y}{\partial y_1}\right] dt \\ &= \int_0^1 \left[ty\frac{\partial L}{\partial y} + tL - tx\frac{\partial M}{\partial y}\right] dt\end{aligned}$$

and similarly,

$$\frac{\partial Y}{\partial z_1} = \int_0^1 \left[tx\frac{\partial N}{\partial z} - tz\frac{\partial L}{\partial z} - tL\right] dt$$

Therefore,

$$\frac{\partial z}{\partial y_1} - \frac{\partial Y}{\partial z_1} = \int_0^1 \left[2tL - tx\left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}\right) + ty\frac{\partial L}{\partial y} + tz\frac{\partial L}{\partial z}\right] dt.$$

Since $\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0$, we can write

$$\frac{\partial z}{\partial y_1} - \frac{\partial Y}{\partial z_1} = \int_0^1 \left[2tL + tx\frac{\partial L}{\partial x} + ty\frac{\partial L}{\partial y} + tz\frac{\partial L}{\partial z} \right] dt.$$

Also,

$$\begin{aligned} tx\frac{\partial L}{\partial x} + ty\frac{\partial L}{\partial y} + tz\frac{\partial L}{\partial z} &= t\left(x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} + z\frac{\partial L}{\partial z} \right) \\ &= t\left(x_1 t\frac{\partial L}{\partial x} + y_1 t\frac{\partial L}{\partial y} + z_1 t\frac{\partial L}{\partial z} \right) \\ &= t^2\left(\frac{\partial L}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial L}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial L}{\partial z}\frac{\partial z}{\partial t} \right) \\ &= t^2\frac{\partial L}{\partial t} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\partial z}{\partial y_1} - \frac{\partial Y}{\partial z_1} &= \int_0^1 \left(t^2\frac{\partial L}{\partial t} + 2tL \right) dt = \int_0^1 \frac{\partial}{\partial t}(t^2 L)\ dt \\ &= t^2 L\Big|_0^1 = L(x_1, y_1, z_1) \end{aligned}$$

This proves the first equivalence; the other two are proved similarly. □

HOMEWORK FOR DAY 52. Page 330 #2 and:
Which of the following are independent of path: page 312 #1 a, b, d, e, Page 330 #1 a, b ?

HOMEWORK ANSWERS. #2 a) $xyz\Big|_{(1,1,2)}^{(3,5,0)} = 0 - 2 = -2$

#2 b) $x \sin yz\Big|_{(1,0,0)}^{(1,0,2\pi)} = 0 - 0 = 0$

53 Section 8.1, Complex Functions

Objective. *Students will understand basics of complex numbers and functions. Students will evaluate functions at complex points.*

Notations For $z = x + iy$, we have $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ as the real and imaginary parts of z.

Conjugate If $z = x + iy$, then the conjugate is $\overline{z} = x - iy$; i.e., $\operatorname{Re} z = \operatorname{Re} \overline{z}$ and $\operatorname{Im} z = -\operatorname{Im} \overline{z}$.

Magnitude If $z = x + iy$, then the magnitude (or absolute value) is $|z| = \sqrt{x^2 + y^2} = r$

Argument If $x = x + iy$, then the argument is $\arg z = \arctan \frac{y}{x} = \theta$.

Triangle Inequality $|z_1 + z_2| \leq |z_1| + |z_2|$

Complex Functions
Denoted $w = f(z)$, we have the following functions.

Polynomials $w = a_0 z^n + \cdots + a_{n-1} z + a_n$

Rationals $w = \dfrac{a_0 z^n + \cdots + a_{n-1} z + a_n}{b_0 z^m + \cdots + b_{m-1} z + b_m}$

Exponentials $w = \exp z = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

Trigonometric $w = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}, \quad w = \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Hyperbolic $w = \sinh z = \dfrac{e^z - e^{-z}}{2}, \quad w = \cosh z = \dfrac{e^z + e^{-z}}{2}$

Logarithmic, inverse trigonometric, and inverse hyperbolic come later.

Example 53.1. *Evaluate* $\exp(2 + i\frac{\pi}{2})$, $\sin(\frac{\pi}{2} - 2i)$, $\tan i$, $\exp(i\pi)$

$$\begin{aligned}
\exp(2+i\tfrac{\pi}{2}) &= e^2(\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}) = e^2 i \\
\sin(\tfrac{\pi}{2}-2i) &= \frac{\exp(2-i\frac{\pi}{2})-\exp(-2+i\frac{\pi}{2})}{2i} \\
&= \frac{e^2(\cos-\frac{\pi}{2}+i\sin-\frac{\pi}{2})-e^{-2}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})}{2i} \\
&= \frac{-e^2 i - e^{-2} i}{2i} = \frac{-e^4-1}{2e^2} \\
\tan i &= \frac{\sin i}{\cos i} = \frac{e^{-1}-e}{2i}\cdot\frac{2}{e^{-1}+e} = \frac{1-e^2}{i(1+e^2)} \\
\exp(i\pi) &= \cos\pi + i\sin\pi = -1
\end{aligned}$$

Example 53.2. *Problems 1 and 2 a b c d e on Page 536*

HOMEWORK FOR DAY 53. Page 536 #2 parts f and g, #3 part a

HOMEWORK ANSWERS. #2 f)

$$\sin iz = \frac{e^{-z} - e^{z}}{2i} = \frac{i}{i} \cdot \frac{e^{-z} - e^{z}}{2i} = \frac{i(e^{z} - e^{-z})}{2} = i \sinh z$$
$$\cos iz = \frac{e^{-z} + e^{z}}{2} = \frac{e^{z} - e^{-z}}{2} = \cosh z$$

#2 g)

$$\overline{e^{z}} = e^{x}(\cos y - i \sin y) = e^{x-iy} = e^{\overline{z}}$$
$$\overline{\sin z} = \frac{\overline{e^{iz}} - \overline{e^{-iz}}}{2i} = \frac{e^{-i\overline{z}} - e^{i\overline{z}}}{-2i} = \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} = \sin \overline{z}$$
$$\overline{\cos z} = \frac{\overline{e^{iz}} + \overline{e^{-iz}}}{2} = \frac{e^{-i\overline{z}} + e^{i\overline{z}}}{2} = \frac{e^{i\overline{z}} + e^{-i\overline{z}}}{2} = \cos \overline{z}$$

#3 a) Since $e^{z} = e^{x}(\cos y + i \sin y)$, we have that either $e^{x} = 0$ or $\cos y + i \sin y = 0$. But we know that e^{x} for real x can never be zero, and $\cos y + i \sin y$ will only be zero when $\cos y = \sin y = 0$, which can never happen.

54 Section 8.2, Complex-Valued Functions of a Real Variable

Objective. *Students will graph, differentiate, and integrate complex functions of a real variable.*

Consider the function $w = F(t) = t + it^2$. This is a function that gives complex values for real-valued inputs. In effect, we have $F(t)$ representable as the sum of two real-valued functions: $f(t) = t$ and $g(t) = t^2$, so that $F(t) = f(t) + ig(t)$.

Note that we can graph these functions by treating them as parametric; i.e., if $F(t) = e^{it}$, then we graph $x(t) = \operatorname{Re} F(t) = \cos t$ and $y = \operatorname{Im} F(t) = \sin t$ and we see that the graph of F is a circle of radius 1 in the complex plane.

Under this interpretation, all theory of limits, continuity, and sums, products, and quotients of functions is similar to real-valued function theory, with the following exceptions.

Limits as $t \to \infty$ is defined as for real functions; however, $\lim_{t\to t_0} F(t) = \infty$ is defined to mean $\lim_{t\to t_0} |F(t)| = \infty$; *there is no concept of* ∞ *for complex numbers.*

Continuity: If $f(t)$ and $g(t)$ are continuous in real numbers, then $F(t) = f(t) + ig(t)$ is continuous in complex numbers.

By using the definition of the derivative on the real and imaginary parts, we have that $F'(t_0) = f'(t_0) + ig'(t_0)$. All derivative rules apply.

The definite integral is done by integrating real and imaginary parts:

$$\int_\alpha^\beta F(t)\ dt = \int_\alpha^\beta f(t)\ dt + i\int_\alpha^\beta g(t)\ dt.$$

If the indefinite integral of $F(t)$ is $G(t)$, then we also have that

$$\int_\alpha^\beta F(t)\ dt = G(\alpha) - G(\beta),$$

and the Fundamental Theorem still holds.

We also have the following basic inequality, where $|F(t)| < M$ on the interval $\alpha \le t \le \beta$:

$$\left|\int_\alpha^\beta F(t)\ dt\right| \le \int_\alpha^\beta |F(t)|\ dt \le M(\beta - \alpha).$$

Example 54.1. *Find* $\int_0^2 (t^2+i)\ dt$.

$$\int_0^2 (t^2+i)\ dt = \left.\frac{t^3}{3} + it\right|_0^2 = \frac{8}{3} + 2i$$

Example 54.2. *Find* $\int_0^1 ie^{(2+i)t}dt$.

$$\int_0^1 ie^{(2+i)t}dt = \left.\frac{ie^{(2+i)t}}{2+i}\right|_0^1 = \frac{i(e^{2+i}-1)}{2+i} = \frac{(1+2i)(e^{2+i}-1)}{4}$$

HOMEWORK FOR DAY 54. Page 536, #4, #5, #7

HOMEWORK ANSWERS. #4

a) A line with slope -1 and intercepts $(0,2)$ and $(2,0)$

b) The top half of the parabola $y = 1 + \sqrt{x}$.

c) Write as $\cos 3t + i \sin 3t$; we have a circle of radius 1 centered at the origin.

d) Write as $2e^{-t}(\cos 2t + i \sin 2t)$; we have a spiral.

e) Write as $te^{-t}(\cos 2t + i \sin 2t)$; we have a looping spiral.

f) Write as $e^{-t} + \sin t - i \cos t$; we have a really weird graph!

#5 and #7: See the textbook

55 Sections 8.3 and 8.4, Complex-Valued Functions of a Complex Variable and Thier Derivatives

Objective. *Students will determine limits and continuity of complex functions of a complex variable. Students will find derivatives of complex functions of a complex variable.*

We consider a function $w = f(z)$ for complex z. We may also write $w = u + iv,\ z = x + iy$ so that we have $u + iv = f(x + iy)$; i.e., if $f(z) = z^2$ for all z, then $f(x + iy) = x^2 - y^2 + 2ixy$ so that $u = x^2 - y^2,\ v = 2xy$.

Every complex function $w = f(z)$ is representable as two real functions:

$$u = u(x, y) = \operatorname{Re} f(z), \quad v = v(x, y) = \operatorname{Im} f(z),$$

and vice versa; i.e., if $u = xy$ and $v = x + y$, then this is equivalent to the complex function $w = xy + i(x + y)$.

Real function decompositions for $e^z,\ \sin z,\ \cos z,\ \sinh z,\ \cosh z$ (Page 537)

Definitions:

Let D be a domain in the complex plane. Let $z_0 \in D$ and let $|z - z_0| < \delta$ be circular neighborhood or radius δ. If $f(z)$ is defined in this neighborhood (except possibly at z_0), then $\lim_{z\to z_0} f(z) = w_0$ if for every $\epsilon > 0$ we have that $|f(z) - w_0| < \epsilon$ for $0 < |z - z_0| < \delta$.

If $f(z_0)$ is defined and $\lim_{z\to z_0} f(z) = w_0 = f(z_0)$, then $f(z)$ is continuous at $f(z_0)$.

Theorem 55.1. *The function $w = f(z)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$ are continuous at (x_0, y_0).*

Theorem 55.2. *The sum, product, quotient, and composition of continuous functions is continuous (except division by zero).*

Thus, all polynomials are continuous, as are e^z, $\cos z$, $\sin z$.

Finally, we write $\lim_{z\to z_0} f(z) = \infty$ if $\lim_{z\to z_0} |f(z)| = \infty$; in other words, "approaching infinity" means moving away from the origin.

Let $w = f(z)$ be given in D and let $z_0 \in D$. Then

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$$

is the derivative of w. This has all the same properties as the real variable derivative—with a few more. This is because $\Delta z \to 0$ in any manner whatever. If we restrict Δz to approach zero along a certain path, then this is equivalent to a "directional derivative." However, the derivative has the same value in any direction.

Theorem 55.3. *If $w = f(z)$ has a differential $dw = c\Delta z$ at z_0, then w has a derivative $f'(z_0) = c$. If w has a derivative at z_0, then w has a differential $dw = f'(z_0)\Delta z$ at z_0.*

As with reals, differentiability implies continuity.

Power, product, quotient, and chain rules all apply.

HOMEWORK FOR DAY 55. Page 540 #1, #2, #3

HOMEWORK ANSWERS. #1

a) $u = x^2 - 2xy - y^2,\ v = x^2 + 2xy - y^2$; continuous on $\mathbb{C}$

b) $u = \dfrac{x^2 + y^2 + y}{x^2 + (y+1)^2},\ v = \dfrac{x}{x^2 + (y+1)^2}$; continuous on $\{z|z \neq -1\}$

c) $u = \dfrac{2e^{2y}\sin 2x}{e^{4y} + 2e^{2y}\cos 2x + 1},\ v = \dfrac{e^{4y} - 1}{e^{4y} + 2e^{2y}\cos 2x + 1}$; continuous except where z is an odd multiple of $\pi/2$.

d) See textbook for the rest

#2, #3: See textbook

56 Section 8.5, Integrals

Objective. *Students will evaluate integrals of complex functions by evaluating the two associated real-valued line integrals.*

A complex integral is defined as a line integral. Let $C : x(t), y(t)$ for $a \leq t \leq b$ be a path in the complex plane from A to B. Subdivide $[a, b]$ and choose $t_j^* \in [t_{j-1}, t_j]$ and set $z_j^* = x(t_j^*) + iy(t_j^*)$. Then, if $f(z) = u(x, y) + iv(x, y)$, we have

$$\begin{aligned}
\int\limits_C f(z)\ dz &= \lim_{\substack{n\to\infty \\ \max\Delta_j t \to 0}} \sum_{j=1}^{n} f(z_j^*)\Delta_j z \\
&= \lim_{\substack{n\to\infty \\ \max\Delta_j x \to 0 \\ \max\Delta_j y \to 0}} \sum_{j=1}^{n} (u + iv)(\Delta_j x + i\Delta_j y) \\
&= \lim_{\substack{n\to\infty \\ \max\Delta_j x \to 0 \\ \max\Delta_j y \to 0}} \sum_{j=1}^{n} (u\Delta_j x - v\Delta_j y) + i\sum_{j=1}^{n} (v\Delta_j x + u\Delta_j y) \\
&= \int\limits_C (u\ dx - v\ dy) + i\int\limits_C (v\ dx + u\ dy)
\end{aligned}$$

So a complex integral is the sum of two real line integrals.

Theorem 56.1. *Let $f(z) \in D$ be continuous, let C be the curve defined by $x(t), y(t), t \in [a, b]$, and let $\frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt}i$. Then $\int_C f(z)\ dz$ exists and*

$$\int\limits_C f(z)\ dz = \int_a^b f(z(t))\frac{dz}{dt}\ dt.$$

Example 56.1. *Page 542, examples 1 and 2*

Normal properties of integrals hold: sum, constant, sum of paths, direction reversal.

Theorem 56.2. *Let $f(z)$ be continuous on C, let $|f(z)| \leq M$ on C, and let $L = \int_C ds = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt$ be the length of C. Then*

$$\left|\int\limits_C f(z)\ dx\right| \leq \int\limits_C |f(z)|\ ds \leq M \cdot L.$$

Example 56.2. *Write $\int_C \sin z \; dz$ as two real line integrals, then show that the integral is independent of path.*

Since $\sin z = \sin x \cosh y + i \cos x \sinh y$, we have

$$\int\limits_C \sin z \; dz =$$

$$\int\limits_C \sin x \cosh y \; dx - \cos x \sinh y \; dy + \int\limits_C \cos x \sinh y \; dx + \sin x \cosh y \; dy$$

Checking, we see that $\frac{\partial u}{\partial y} = \sin x \sinh y = \frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$.

Example 56.3. *Evaluate $\oint \frac{1}{z} \; dz$ on the circle $|z| = R$.*

Since the circle can be expressed as $z = Re^{it}$, we have

$$\begin{aligned}\oint \frac{1}{z} \; dz &= \int_0^{2\pi} \frac{1}{Re^{it}} Rie^{it} \; dt \\ &= \int_0^{2\pi} i \; dt = 2i\pi\end{aligned}$$

Example 56.4. *Prove that $\oint \frac{1}{z} \; dx = 0$ on every simple closed path not meeting or enclosing the origin.*

Note that if C does enclose the origin, we get a nonzero value by the previous example. So let C be the simple closed path (x, y) for $t \in [a, b]$ where C does not meet or enclose the origin. Then $(x(a), y(a)) = (x(b), y(b))$ and

$$\begin{aligned}\oint \frac{1}{z} \; dz &= \int_a^b \frac{1}{x + iy} \left(\frac{dx}{dt} + i \frac{dy}{dt} \right) dt \\ &= \log(x + iy)\Big|_a^b \\ &= \log[x(b) + iy(b)] - \log[x(a) + iy(a)] \\ &= 0\end{aligned}$$

HOMEWORK FOR DAY 56. Page 544 #1, #2 parts a and b, #3 part c

HOMEWORK ANSWERS. #1

a) $C : (x, y) = (t, t)$ for $t \in [0, 1]$. Thus,

$$\int_C (x^2 - iy^2)\,dz = \int_0^1 (t^2 - it^2)(1+i)\,dt = (1-i)(1+i)\int_0^1 t^2\,dt = \frac{2}{3}t^3\Big|_0^1 = \frac{2}{3}$$

b) $C : (x, y) = (t, \sin t)$ for $t \in [0, \pi]$. Thus,

$$\begin{aligned}\int_C z\,dz &= \int_0^\pi (t + i\sin t)(i + i\cos t)\,dt \\ &= \int_0^\pi (t - \sin t\cos t + i[t\cos t + \sin t])\,dt \\ &= \frac{1}{2}t^2 - \frac{1}{4}\cos 2t + it\sin t\Big|_0^\pi \\ &= \frac{1}{2}\pi^2 - \frac{1}{4} + \frac{1}{4} = \frac{1}{2}\pi^2\end{aligned}$$

c) $C : (x, y) = (1, t)$ for $t \in [0, 1]$. Then,

$$\int_C \frac{1}{z}\,dz = \int_0^1 \frac{1}{1+it} i\,dt = \ln|1 + it|\big|_0^1 = \ln(1 + i)$$

#2

a) $z + 1 = x + 1 + iy$ so $u = x + 1,\ v = y$. Then,

$$\int_C (z + 1)\,dz = \int_C (x + 1)\,dx - y\,dy + i\int_C y\,dx + (x + 1)\,dy$$

and $\frac{\partial}{\partial y}(x + 1) = \frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial y} = \frac{\partial}{\partial x}(x + 1) = 1$.

b)