Calculus As Problem Solving

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October 3, 2008

The approach to calculus as problem solving is two-fold:

- develop problem-solving techniques within the calculus milieu, and
- use calculus as a problem-solving technique.

To this end, we present a few topics and some problems that are indicative of a problem-solving approach to calculus.

Euler's method and slope fields are not, in terms of concepts or calculations, calculus concepts *per se*. One can compute an Euler's method approximation and graph a slope field without knowing about derivatives or integrals. Phrased in terms of "rate equations," one can develop the algorthmic and conceptual framework of Euler's method, and from there, introduce Riemann sums. One cannot *solve* rate equations without knowledge of differentiation and integration, but this is not the point of Euler's method anyway.

1 Euler's Method Equals Riemann Sums

The defining (recursive) equation for Euler's Method on the rate equation $y' = f(x, y)$ with initial value $y(x_0) = y_0$ with step size Δx is

$$y(x_{n+1}) = y(x_n) + y'(x_n)\Delta x.$$

We approximate, say, $f(x_4)$ by using Euler's method four times:

$$\begin{aligned}
y(x_1) &= y(x_0) + y'(x_0)\Delta x \\
y(x_2) &= y(x_1) + y'(x_1)\Delta x \\
y(x_3) &= y(x_2) + y'(x_2)\Delta x \\
y(x_4) &= y(x_3) + y'(x_3)\Delta x.
\end{aligned}$$

Seconds	Velocity (ft/sec)	Seconds	Velocity (ft/sec)
0	0	8	27.477
1	3.494	9	30.837
2	6.971	10	34.180
3	10.431	11	37.506
4	13.874	12	40.816
5	17.300	13	44.110
6	20.709	14	47.388
7	24.101	15	50.650

Table 1: VELOCITIES OF A BARREL OF NUCLEAR WASTE

However, a simple rewriting of each equation gives us

$$\begin{aligned} y(x_1) - y(x_0) &= y'(x_0)\Delta x \\ y(x_2) - y(x_1) &= y'(x_1)\Delta x \\ y(x_3) - y(x_2) &= y'(x_2)\Delta x \\ y(x_4) - y(x_3) &= y'(x_3)\Delta x. \end{aligned}$$

Now, sum the above equations, and the "inner" function values cancel[1] to give

$$y(x_4) - y(x_0) = \sum_{i=0}^{3} y'(x_i)\Delta x.$$

If we generalize, then we can write

$$y(x_n) - y(x_0) = \sum_{i=0}^{n-1} y'(x_i)\Delta x.$$

Thus, calling $x_0 = a$ and $x_n = b$, if we let $\Delta x \to 0$, we have part of the Fundamental Theorem of Calculus:

$$y(b) - y(a) = \int_a^b y'(x)\, dx.$$

Problem 1. Consider Table 1, which displays the velocity of a barrel of nuclear waste at each second over a 15-second fall through the sea. How far has it fallen over the 15 seconds?

The motivation for the rest of the course is to find *exact solutions* to rate equations.

2 Recognizing Solutions to Rate Equations

Once derivatives and differentiation are introduced, we always connect our ideas to rate equations.

[1]Telescoping series in action!

Problem 2. One of the basic trigonometric identities is the so-called *Pythagorean Identity*:

$$\sin^2 x + \cos^2 x = 1.$$

Solving this identity for $\cos x$, we have

$$\cos x = \sqrt{1 - \sin^2 x}.$$

Thus, if $y = \sin x$, then

$$y' = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}.$$

Notice that this is now a rate equation

$$y' = \sqrt{1 - y^2}$$

and we know that the solution to this rate equation must be $y = \sin x$.

(a) Sketch a slope field for this rate equation.

(b) Use Euler's method with initial value $y(0) = 0$ and a step size of 0.25 to approximate $y(1)$.

(c) Use your calculator to find the value of $y(1) = \sin 1$.

(d) Determine a step size so that Euler's method will approximate $\sin 1$ to within 3 decimal places.

(e) Find the solution to the rate equation $y' = -\sqrt{1 - y^2}$.

Problem 3. Notice that if $y = \left(x^2 + 4\right)^5$, then, by the Chain Rule, we have $y' = 10x\left(x^2 + 4\right)^4$. Use this information to solve the rate equation $y' = 10xy^{4/5}$.

Problem 4. Find a solution to the rate equation $y' = 6xy^{2/3}$.

Problem 5. The graph of a function of the form

$$P(t) = \frac{M}{1 + Ce^{-rMt}},$$

where M, r, and C are constants, is a logistic curve. Graph the function

$$y(x) = \frac{8}{1 + 10e^{-0.9x}}$$

in the window $-1 \le x \le 10$, $-1 \le y \le 9$. What value does y approach as $x \to \infty$? What appears to be the y-value of the point where y' is changing the fastest? How are these numbers related to the logistic rate equation?

3 Proof Techniques in Calculus

The common proof technique of *reductio ad absurdum*, or "proof by contradiction," along with the problem-solving technique of "defining a new function" are used to prove important theorems and are to be remembered as useful techniques.

The proof below is based on the that found in Apostol [1967].

Theorem 1 (Rolle's Theorem). *Let $f(x)$ be differentiable on (a,b) and continuous on $[a,b]$ where $f(a)=f(b)$. Then there is some point c in (a,b) such that $f'(c)=0$.*

Proof. We will use the method of proof by contradiction. Assume $f'(x)\neq 0$ for every x in (a,b).

By the Extreme Value Theorem, f has a global maximum M and a global minimum m. Fermat's Test indicates that neither extreme value can be taken in (a,b) or else f' is zero there; hence, both extrema occur at the endpoints of the interval. But since $f(a)=f(b)$, we have $M=m$ and thus f is constant on (a,b). This implies $f'(x)=0$ everywhere on (a,b); thus, we have a contradiction. Hence, there is at least one c in (a,b) such that $f'(c)=0$. □

Next, we prove Cauchy's Mean Value Theorem. The proof relies on the "define a new function" technique. This is a nice result that allows one to trivially prove the standard Mean Value Theorem.

Theorem 2 (Cauchy's Mean Value Theorem). *Let f and g be two functions continuous on $[a,b]$ and differentiable on (a,b). Then, for some c in (a,b),*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$$

provided $g(b)\neq g(a)$ and $g'(c)\neq 0$.

Proof. We note that the quantity

$$k=\frac{f(b)-f(a)}{g(b)-g(a)}$$

is constant since it depends only on the real number constants a and b. We introduce the function $P(x)=f(x)-kg(x)$ on $[a,b]$, where k is defined above. Then we have

$$\begin{aligned}P(x)&=f(x)-kg(x)\\&=f(x)-\frac{f(b)-f(a)}{g(b)-g(a)}g(x)\\&=\frac{f(x)[g(b)-g(a)]-g(x)[f(b)-f(a)]}{g(b)-g(a)}\end{aligned}$$

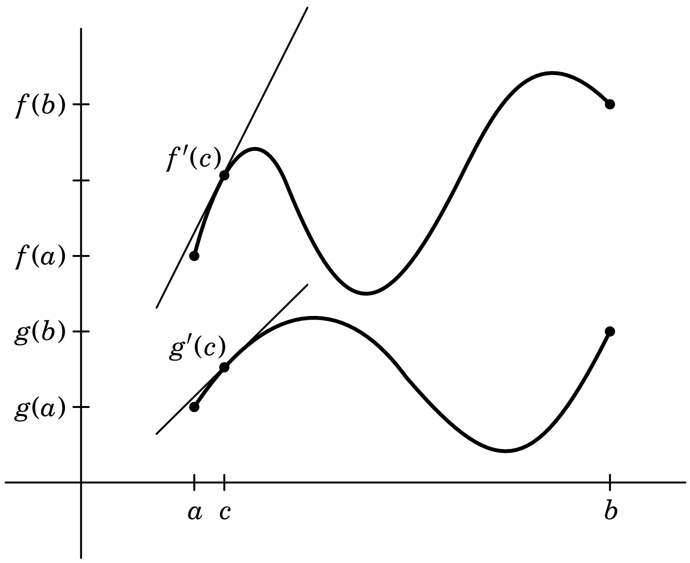

Figure 1: TWO FUNCTIONS DEMONSTRATING CAUCHY'S MEAN VALUE THEOREM

so that

$$\begin{aligned} P(a) &= f(a) - kg(a) \\ &= \frac{f(a)[g(b)-g(a)] - g(a)[f(b)-f(a)]}{g(b)-g(a)} \\ &= \frac{f(a)g(b) - g(a)f(b)}{g(b)-g(a)} \end{aligned}$$

and

$$\begin{aligned} P(b) &= f(b) - kg(b) \\ &= \frac{f(b)[g(b)-g(a)] - g(b)[f(b)-f(a)]}{g(b)-g(a)} \\ &= \frac{-f(b)g(a) + g(b)f(a)}{g(b)-g(a)} \end{aligned}$$

implying that $P(a) = P(b)$. Hence, we may apply Rolle's Theorem. Therefore, there exists a point c in (a,b) such that $P'(c) = 0$. We have

$$\begin{aligned} P'(c) = f'(c) - kg'(c) &= 0 \\ f'(c) &= kg'(c) \\ \frac{f'(c)}{g'(c)} &= k \\ \frac{f'(c)}{g'(c)} &= \frac{f(b)-f(a)}{g(b)-g(a)} \end{aligned}$$

which proves the theorem. $\square$

In the course of developing the methods and techniques, we define the logarithm function $\ln(x) = L(x)$ as

$$L(x) = \int_1^x \frac{1}{t}\,dt.$$

Properties of this function are fully investigated, including all standard logarithm properties.

When it comes to the exponential function $\exp(x) = E(x)$, we define it as the inverse of $L(x)$. As the chain rule and the Fundamental Theorem has been well-established at this point, it is now very easy to prove the following.

Theorem 3. *The derivative of* $\exp(x)$ *is itself.*

Proof. To determine the derivative, we will begin by composing $\ln(x)$ with $\exp(x)$ in two ways. First, we have that $\ln(\exp(x)) = x$. Recalling that $\ln(x) = \int_1^x \frac{1}{t}\,dt$, we also have that

$$\ln(\exp(x)) = \int_1^{\exp(x)} \frac{1}{t}\,dt.$$

Therefore,

$$\int_1^{\exp(x)} \frac{1}{t}\,dt = x.$$

Taking derivatives of both sides, we get

$$\frac{1}{\exp(x)} \cdot \exp'(x) = 1, \qquad \text{or} \qquad \exp'(x) = \exp(x). \qquad \square$$

4 Illustrative Problems

These problems are to be done without a calculator.

Problem 6. Let $f(x) = |x| + x$. Does $f'(0)$ exist? Explain.

Problem 7. (Taylor and Mann [1983]) Let $f(x) = 5\sqrt{16+x^2} + 4\sqrt{(3-x)^2}$, where we take the positive root so that $\sqrt{(3-x)^2} = |3-x|$. Note that f is differentiable except when $x = 3$ and that $f(x) \to \infty$ as $x \to \pm\infty$.

(a) How do you infer from this information that f must attain a global minimum value?

(b) Find formulas for $f'(x)$ when $x < 3$ and when $x > 3$, and show that $f' < 0$ if $x < 3$ and that $f' > 0$ if $x > 3$.

(c) Is the derivative continuous at $x = 0$? What happens to the graph of f at $x = 0$?

(d) What is the minimum value of f?

Problem 8. (Fraga [1993]) In medicine, the reaction $R(x)$ to a dose x of a drug is given by $R(x) = Ax^2(B - x)$, where $A > 0$ and $B > 0$. The sensitivity $S(x)$ of the body to a dose of size x is defined to be $R'(x)$. Assume that a negative reaction is a bad thing.

(a) What seems to be the domain of R? What seems to be the physical meaning of the constant B? What seems to be the physical meaning of the constant A?

(b) For what value of x is R a maximum?

(c) What is the maximum value of R?

(d) For what value of x is the sensitivity a minimum?

(e) Why is it called sensitivity?

Problem 9. (Zeitz [1999]) For some constant $a > 1$, consider the convergent sequence $\{x_n\}$ defined by $x_0 = a$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

for $n = 1, 2, \ldots$. To what does this sequence converge?

Problem 10. (AP) Suppose that $5x^3 + 40 = \int_c^x f(t)\,dt$.

(a) What is $f(x)$?

(b) Find the value of c.

Problem 11. (Fraga [1993]) Let $G(x) = \int_0^x \sqrt{16 - t^2}\,dt$.

(a) Find $G(0)$.

(b) Does $G(2) = G(-2)$? Does $G(2) = -G(-2)$?

(c) What is $G'(2)$?

(d) What are $G(4)$ and $G(-4)$?

Problem 12. (Roberts [1996]) Table 2 shows four points on the function f.

(a) Estimate $f'(10)$.

(b) Estimate $f^{-1}(8.5)$

(c) Estimate $\int_8^{10} f'(t)\,dt$.

Problem 13. (Stein and Barcellos [1992]) When computing the internal energy of a crystal, Claude Garrod, in his book *Twentieth Century Physics* [1984], states that the integral

$$\int_0^{\pi/2} \frac{\sin x}{e^{0.26 \sin x} - 1}\,dx$$

"cannot be evaluated analytically. However, it can easily be computed numerically using Simpson's rule. The result is 5.56."

t	$f(t)$
8	8.253
9	8.372
10	8.459
11	8.616

Table 2: DATA FOR PROBLEM 12

(a) Is the integral proper or improper? Why?

(b) What is the limit of the integrand as $x \to 0^+$?

(c) What does "cannot be evaluated analytically" mean?

(d) Is it possible to use your calculator program to approximate the integral by Simpson's rule with $n = 6$? If so, approximate it to four decimal places; if not, why not?

Problem 14. (Fraga [1993]) Four calculus students disagree as to the value of the integral $\int_0^\pi \sin^8 x\, dx$. Abby says that it is equal to π. Nika says that it is equal to $35\pi/128$. Catherine claims it is equal to $3\pi/90 - 1$, while Peyton says its equal to $\pi/2$. One of them is right. Which one is it?

Problem 15. (Zeitz [1999]) Compute $\displaystyle\int_0^{\pi/2} \cos^2 x\, dx$ in your head.

Problem 16. (Fraga [1993]) Show that the region enclosed by the x-axis and the graph of the parabola

$$f(x) = \frac{2}{a^2}x - \frac{1}{a^3}x^2, \quad a > 0$$

has an area that is independent of the value of a. How large is this area? What curve is determined by the vertices of all these parabolas?

Problem 17. *Partial Fractions Versus Trig Substitution*

(a) Graph the function $f(x) = \frac{1}{x^2-4}$ on your paper.

(b) Is the definite integral $\int_{-1}^{1} \frac{dx}{x^2-4}$ negative or positive? Justify your answer with reference to your graph.

(c) Compute the integral in part (b) by using partial fractions.

(d) A Georgia Tech calculus student suggests instead to use the substitution $x = 2\sec\theta$. Compute the integral in this way, or describe why this substitution fails.

Problem 18. (AP) If we use the substitution $u = x/2$ in the integral

$$\int_2^4 \frac{1-(x/2)^2}{x}\,dx,$$

then the integral becomes

A) $\int_1^2 \frac{1-u^2}{u}\,du$ B) $\int_2^4 \frac{1-u^2}{u}\,du$ C) $\int_2^4 \frac{1-u^2}{4u}\,du$

D) $\int_2^4 \frac{1-u^2}{2u}\,du$ E) $\int_1^2 \frac{1-u^2}{2u}\,du$

Problem 19. (AP) If

$$\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2 \cos x\,dx,$$

then $f(x)$ could be

A) $3x^2$ B) x^3 C) $-x^3$ D) $\sin x$ E) $\cos x$

Problem 20. (Fraga [1993]) Justin and Jonathan are having an argument as to the value of $\int \sec^2 x \tan x\,dx$. Justin makes the substitution $u = \sec x$ and gets the answer $\frac{1}{2}\sec^2 x$, whereas Jonathan makes the substitution $u = \tan x$ and gets the answer $\frac{1}{2}\tan^2 x$. Please get them to stop arguing by explaining to them why their antiderivatives are both acceptable.

Problem 21. (Finney et al. [2001]) You are driving along the highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant decceleration is required to stop your car in 242 ft?

Problem 22. Let f and g be continuous and differentiable functions satisfying the given conditions for some real number B:

I. $\int_1^3 f(x+2)\,dx = 3B$
II. The average value of f in the interval $[1,3]$ is $2B$
III. $\int_{-4}^x g(t)\,dt = f(x) + 3x$
IV. $g(x) = 4B + f'(x)$

(a) Find $\int_1^5 f(x)\,dx$ in terms of B.

(b) Find B.

Problem 23. (Finney et al. [2001]) Solve the differential equation $dP/dt = kP^2$ for constant k, with initial condition $P(0) = P_0$. Prove that the graph of the solution has a vertical asymptote at a positive value of t. What is that value of t? (This value is called the *catastrophic solution*.)

Problem 24. A particle is moving along the path of the curve determined by the parametric equations $x = e^{2t} + 1$ and $y = \ln(e^{4t} + 2e^{2t} + 1)$, where $t > 0$.

(a) Find dy/dx in terms of t, then find the equation of the tangent line at time $t = \frac{1}{2}$.

(b) Show that $\dfrac{d^2y}{dx^2} = \dfrac{-2}{(e^{2t}+1)^2}$.

(c) Sketch the path of the curve and indicate the direction of motion.

(d) Write the set of parametric equations without the parameter t.

Problem 25. (1990 Putnam Exam) Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x \left[(f(t))^2 + (f'(t))^2\right]\,dt + 1990.$$

Problem 26. (Roberts [1996]) During your teacher's days as a student last century, he often studied calculus in a dim unheated room with only one candle for light and heat. One particular day in mid-winter, after walking 10 miles (uphill both ways!) through knee-deep snow to attend class, he returned home too tired to study. After lighting the solitary candle on his desk, he walked directly away cursing his woeful situation. The temperture (in degrees Fahrenheit) and illumination (in percentage of candle-power) decreased as his distance (in feet) from his candle increased. In fact, he kept a record of this and in Table 3 is that information, just in case you may not believe the preceding sad tale!

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

a) What is the average rate at which the temperature is changing when the illumination drops from 77% to 56%?

b) I can still read my old unlit analog watch when the illumination is 64%. Can I still read my watch when I am 3.5 feet from the candle? Explain.

c) Suppose that at 6 feet the instantaneous rate of change of the temperature is -4.5°F per foot and the instantaneous rate of change of the illumination is -3% candle-power per foot. Estimate the temperature and the illumination at 7 feet.

d) Am I in the dark before I am cold or am I cold before I am in the dark? Explain.

Problem 27. (Zeitz [1999]) Compute $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{n}{k^2+n^2}$.

Problem 28. (Roberts [1996]) Jay is a waiter at a fine-dining restaurant with 100 tables. During his first month he waited on 20 tables every night, and collected an average tip of $15 from each table. He started to work more tables, and noticed that for every extra table he took on in a night, his average tip would go down 25 cents per table. He figures that he is physically capable of waiting on up to 30 tables in a night. If Jay wants to maximize his tip money, how many more tables should he wait on?

Distance (feet)	Temperature (°F)	Illumination (% candle-power)
0	55.0	100
1	54.5	88
2	53.5	77
3	52.0	68
4	50.0	60
5	47.0	56
6	43.5	53

Table 3: TABLE FOR PROBLEM 26

Problems For You to Try

Problem 29. (Palmaccio [1996]) A builder is purchasing a rectangular plot of land with frontage on a road for the purpose of constructing a rectangular warehouse. Its floor area must be 300,000 square feet. Local building codes require that the building be set back 40 feet from the road and that there be empty buffer strips of land 25 feet wide on the sides and 20 feet wide in the back. Find the overall dimensions of the parcel of land and building which will minimize the total area of the land parcel that the builder must purchase.

Problem 30. Suppose $f(x)$ is a continuous increasing function for all positive real numbers x. Let $a > 0$. Then further suppose that t is the tangent line of $f(x)$ at the point $(a, f(a))$, and that n is the normal line of $f(x)$ at the point $(a, f(a))$. Let t intersect the x-axis at $(x_t, 0)$ and n intersect the x-axis at $(x_n, 0)$. Let R represent the area of the triangle formed by connecting the points $(a, f(a))$, $(x_t, 0)$, and $(x_n, 0)$.

(a) Find an expression, in terms of $f(a)$ and $f'(a)$, for R.

(b) Let $f(x) = x^3 + 1$ and $a = 1$. Find intersection coordinates x_t and x_n, and then find area of the triangle formed by the points $(1, f(1))$, $(x_t, 0)$, and $(x_n, 0)$.

(c) What would be different about your expression in part (a) if $f(x)$ was a continuous decreasing function?

Problem 31. Let

$$f(x) = \int_0^{x^2} \frac{\sqrt{1+t^2}}{2\sqrt{t}}\, dt.$$

Without using a calculator, find any extreme points and inflection points of $f(x)$, then sketch its graph.

Problem 32. (Smith and Moore [1996]) The cooling system in my old truck holds about 10 liters of coolant. Last summer, I flushed the system by running tap water into a tap-in on the heater hose while the engine was running and simultaneously draining the thoroughly mixed fluid from the bottom of the radiator. Water flowed in at the same rate that the mixture flowed out – at about 2 liters per minute. The system was initially 50% antifreeze. If we let W be the amount of water in the system after t minutes, then it follows that

$$\frac{dW}{dt} = 2 - 2\left(\frac{W}{10}\right).$$

(a) Explain why the equation above is the correct one.

(b) Find W as a function of time.

(c) How long should I have let water run into the system to ensure that 95% of the mixture was water?

Problem 33. (AP) $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points on the curve $y = ax^2 + bx + c$, with $a \neq 0$. A line is drawn parallel to the chord P_1P_2 and tangent to the curve at the point $P(x_0, y_0)$. Prove that $x_0 = (x_1 + x_2)/2$.

Problem 34. (AP) Given a function $f(x)$ defined for all real x, and such that $f(x+h) - f(x) < 6h^2$ for all real h and x. Show that $f(x)$ is a constant.

Problem 35. (Zeitz [1999]) Let $f(x) = \prod_{k=0}^{n}(x+k)$. Find $f'(1)$.

Problem 36. (Taylor and Mann [1983]) A well is located at the point $(0, a)$, and a house is located at $(b, 0)$, where $a, b > 0$. A pipline is to be laid in two straight parts, the first part from the spring to $(x, 0)$, and the second part from $(x, 0)$ to the house. The two parts will cost c_1 and c_2 dollars per unit length, respectively.

(a) Show that the total cost of the pipeline, for *any* value of x, is

$$f(x) = c_1\sqrt{a^2 - x^2} + c_2|b - x|.$$

(b) Show that f has its global minimum value for some x such that $0 < x \leq b$. (*Hint:* Consider the sign of $f'(x)$ when $x \leq 0$ and when $x > b$.)

(c) Find the inequalities which must be satisfied by c_1, c_2, a, and b if f is to attain its minimum for $0 < x < b$.

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