# Calculus As Problem Solving

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# Outline

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Euler's Method Equals Riemann Sums

Solutions to Rate Equations

Proof Techniques

Illustrative Problems

# Euler's Method

The defining (recursive) equation for Euler's Method on the rate equation $y' = f(x,y)$ with initial value $y(x_0) = y_0$ with step size $\Delta x$ is

$$y(x_{n+1}) = y(x_n) + y'(x_n)\Delta x.$$

# Using Euler's Method

We approximate $f(x_4)$ by using Euler's method four times:

$$
\begin{aligned}
y(x_1) &= y(x_0) + y'(x_0)\Delta x \\
y(x_2) &= y(x_1) + y'(x_1)\Delta x \\
y(x_3) &= y(x_2) + y'(x_2)\Delta x \\
y(x_4) &= y(x_3) + y'(x_3)\Delta x.
\end{aligned}
$$

A simple rewriting of each equation gives us

$$
\begin{aligned}
y(x_1) - y(x_0) &= y'(x_0)\Delta x \\
y(x_2) - y(x_1) &= y'(x_1)\Delta x \\
y(x_3) - y(x_2) &= y'(x_2)\Delta x \\
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# Summing Iterations of Euler's Method

Sum the previous equations to get

$$y(x_4) - y(x_0) = \sum_{i=0}^{3} y'(x_i)\Delta x.$$

If we generalize, then we can write

$$y(x_n) - y(x_0) = \sum_{i=0}^{n-1} y'(x_i)\Delta x.$$

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Calling $x_0 = a$ and $x_n = b$, if we let $\Delta x \to 0$, we have

$$y(b) - y(a) = \lim_{\Delta x \to 0} \sum_{i=0}^{n-1} y'(x_i)\Delta x = \int_a^b y'(x)\,dx.$$

# Problem 1

Consider Table 1, which displays the velocity of a barrel of nuclear waste at each second over a 15-second fall through the sea. How far has it fallen over the 15 seconds?

| Seconds | Velocity (ft/sec) | Seconds | Velocity (ft/sec) |
|---|---|---|---|
| 0 | 0 | 8 | 27.477 |
| 1 | 3.494 | 9 | 30.837 |
| 2 | 6.971 | 10 | 34.180 |
| 3 | 10.431 | 11 | 37.506 |
| 4 | 13.874 | 12 | 40.816 |
| 5 | 17.300 | 13 | 44.110 |
| 6 | 20.709 | 14 | 47.388 |
| 7 | 24.101 | 15 | 50.650 |

# Outline

# Problem 2

Beginning with

$$\sin^2 x + \cos^2 x = 1,$$

we solve this identity for $\cos x$ to get

$$\cos x = \sqrt{1 - \sin^2 x}.$$

Thus, if $y = \sin x$, then

$$y' = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}.$$

# Problem 2 continued

So the solution to the rate equation $y' = \sqrt{1-y^2}$ must be $y = \sin x$.

**(a)** Sketch a slope field for this rate equation.

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**(d)** Determine a step size so that Euler's method will approximate $\sin 1$ to within 3 decimal places.

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**(d)** Determine a step size so that Euler's method will approximate $\sin 1$ to within 3 decimal places.

**(e)** Find the solution to the rate equation $y' = -\sqrt{1-y^2}$.

# Problem 3

Notice that if $y = \left(x^2+4\right)^5$, then, by the Chain Rule, we have

$$y' = 10x\left(x^2+4\right)^4.$$

Use this information to solve the rate equation $y' = 10xy^{4/5}$.

# Problem 4

Find a solution to the rate equation $y' = 6xy^{2/3}$.

# Problem 5

The graph of a function of the form

$$P(t) = \frac{M}{1 + Ce^{-rMt}},$$

where $M$, $r$, and $C$ are constants, is a logistic curve. Graph the function

$$y(x) = \frac{8}{1 + 10e^{-0.9x}}$$

in the window $-1 \leq x \leq 10$, $-1 \leq y \leq 9$.

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in the window $-1 \le x \le 10$, $-1 \le y \le 9$. What value does $y$ approach as $x \to \infty$? What appears to be the $y$-value of the point where $y'$ is changing the fastest? How are these numbers related to the logistic rate equation?

# Outline

# Contradiction

**Theorem (Rolle's Theorem)**

*Let $f(x)$ be differentiable on $(a,b)$ and continuous on $[a,b]$ where $f(a) = f(b)$. Then there is some point $c$ in $(a,b)$ such that $f'(c) = 0$.*

# Contradiction

**Proof.**

(Apostol [1967]) We will use the method of proof by contradiction. Assume $f'(x) \neq 0$ for every $x$ in $(a,b)$.
By the Extreme Value Theorem, $f$ has a global maximum $M$ and a global minimum $m$. Fermat's Test indicates that neither extreme value can be taken in $(a,b)$ or else $f'$ is zero there; hence, both extrema occur at the endpoints of the interval. But since $f(a) = f(b)$, we have $M = m$ and thus $f$ is constant on $(a,b)$. This implies $f'(x) = 0$ everywhere on $(a,b)$; thus, we have a contradiction. Hence, there is at least one $c$ in $(a,b)$ such that $f'(c) = 0$. □

# Define a New Function

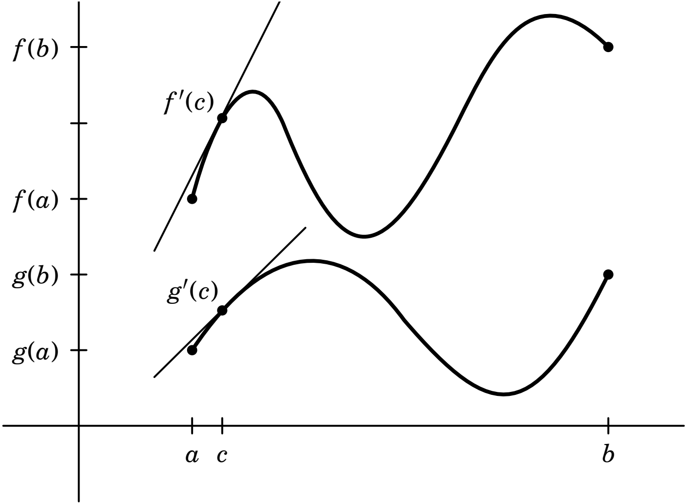
**Theorem (Cauchy's Mean Value Theorem)**

*Let $f$ and $g$ be two functions continuous on $[a,b]$ and differentiable on $(a,b)$. Then, for some $c$ in $(a,b)$,*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

*provided $g(b) \neq g(a)$ and $g'(c) \neq 0$.*

# Figure 1



# Define a New Function

**Proof.**

We note that the quantity

$$k = \frac{f(b) - f(a)}{g(b) - g(a)}$$

is constant since it depends only on the real number constants $a$ and $b$. We introduce the function $P(x) = f(x) - kg(x)$ on $[a, b]$, where $k$ is defined above.

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is constant since it depends only on the real number constants $a$ and $b$. We introduce the function $P(x) = f(x) - kg(x)$ on $[a, b]$, where $k$ is defined above. Then we have $P(a) = P(b)$. Hence, we may apply Rolle's Theorem. Therefore, there exists a point $c$ in $(a, b)$ such that $P'(c) = 0$. We have $P'(c) = f'(c) - kg'(c) = 0$, or $f'(c) = kg'(c)$. Then

$$\frac{f'(c)}{g'(c)} = k = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \square$$

# A Corollary

How should we prove the standard Mean Value Theorem?

# Keep It Simple

We have previously defined

- the logarithm function $\ln(x) = L(x)$ as

$$L(x) = \int_1^x \frac{1}{t}\,dt;$$

- the exponential function $\exp(x) = E(x)$ as the inverse of $L(x)$;
- the chain rule and the Fundamental Theorem.

# Keep It Simple

**Theorem**

*The derivative of* $\exp(x)$ *is itself.*

# Keep It Simple

**Proof.**

We begin by composing $\ln(x)$ with $\exp(x)$ in two ways. First, we have that $\ln(\exp(x)) = x$. Second, we also have that

$$\ln(\exp(x)) = \int_1^{\exp(x)} \frac{1}{t}\, dt.$$

Therefore,

$$\int_1^{\exp(x)} \frac{1}{t}\, dt = x.$$

Taking derivatives of both sides, we get

$$\frac{1}{\exp(x)} \cdot \exp'(x) = 1, \qquad \text{or} \qquad \exp'(x) = \exp(x). \qquad \square$$

# Outline

# Problem 6

Let $f(x) = |x| + x$. Does $f'(0)$ exist? Explain.

# Problem 7

Let $f(x) = 5\sqrt{16+x^2} + 4\sqrt{(3-x)^2}$, where we take the positive root so that $\sqrt{(3-x)^2} = |3-x|$. Note that $f$ is differentiable except when $x = 3$ and that $f(x) \to \infty$ as $x \to \pm\infty$.

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**(a)** How do you infer from this information that $f$ must attain a global minimum value?

**(b)** Find formulas for $f'(x)$ when $x < 3$ and when $x > 3$, and show that $f' < 0$ if $x < 3$ and that $f' > 0$ if $x > 3$.

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**(d)** What is the minimum value of $f$?

# Problem 8

In medicine, the reaction $R(x)$ to a dose $x$ of a drug is given by $R(x) = Ax^2(B - x)$, where $A > 0$ and $B > 0$. The sensitivity $S(x)$ of the body to a dose of size $x$ is defined to be $R'(x)$. Assume that a negative reaction is a bad thing.

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**(e)** Why is it called sensitivity?

# Problem 9

For some constant $a > 1$, consider the convergent sequence $\{x_n\}$ defined by $x_0 = a$ and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

for $n = 1, 2, \ldots$. To what does this sequence converge?

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**(b)** Find the value of $c$.

# Problem 11

Table 2 shows four points on the function $f$.

| $t$ | $f(t)$ |
|---|---|
| 8 | 8.253 |
| 9 | 8.372 |
| 10 | 8.459 |
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Table 2 shows four points on the function $f$.

**(a)** Estimate $f'(10)$.

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**(c)** What is $G'(2)$?

**(d)** What are $G(4)$ and $G(-4)$?

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When computing the internal energy of a crystal, Claude Garrod, in his book *Twentieth Century Physics* [1984], states that the integral

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**(a)** Is the integral proper or improper? Why?

**(b)** What is the limit of the integrand as $x \to 0^+$?

**(c)** What does "cannot be evaluated analytically" mean?

**(d)** Is it possible to use your calculator program to approximate the integral by Simpson's rule with $n = 6$? If so, approximate it to four decimal places; if not, why not?

# Problem 14

Four calculus students disagree as to the value of the integral $\int_0^\pi \sin^8 x\, dx$. Abby says that it is equal to $\pi$. Nika says that it is equal to $35\pi/128$. Catherine claims it is equal to $3\pi/90 - 1$, while Peyton says its equal to $\pi/2$. One of them is right. Which one is it?

# Problem 15

Compute $\int_0^{\pi/2} \cos^2 x \, dx$ in your head.

# Problem 16

Show that the region enclosed by the $x$-axis and the graph of the parabola

$$f(x) = \frac{2}{a^2}x - \frac{1}{a^3}x^2, \quad a > 0$$

has an area that is independent of the value of $a$. How large is this area? What curve is determined by the vertices of all these parabolas?

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Partial Fractions Versus Trig Substitution

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**(c)** Compute the integral in part (b) by using partial fractions.

**(d)** A Georgia Tech calculus student suggests instead to use the substitution $x = 2\sec\theta$. Compute the integral in this way, or describe why this substitution fails.

# Problem 18

If we use the substitution $u = x/2$ in the integral

$$\int_2^4 \frac{1-(x/2)^2}{x}\,dx,$$

then the integral becomes

A) $\displaystyle\int_1^2 \frac{1-u^2}{u}\,du$ B) $\displaystyle\int_2^4 \frac{1-u^2}{u}\,du$ C) $\displaystyle\int_2^4 \frac{1-u^2}{4u}\,du$

D) $\displaystyle\int_2^4 \frac{1-u^2}{2u}\,du$ E) $\displaystyle\int_1^2 \frac{1-u^2}{2u}\,du$

# Problem 19

If

$$\int f(x)\sin x\,dx = -f(x)\cos x + \int 3x^2\cos x\,dx,$$

then $f(x)$ could be

A) $3x^2$ B) $x^3$ C) $-x^3$ D) $\sin x$ E) $\cos x$

# Problem 20

Justin and Jonathan are having an argument as to the value of $\int \sec^2 x \tan x \, dx$. Justin makes the substitution $u = \sec x$ and gets the answer $\frac{1}{2}\sec^2 x$, whereas Jonathan makes the substitution $u = \tan x$ and gets the answer $\frac{1}{2}\tan^2 x$. Please get them to stop arguing by explaining to them why their antiderivatives are both acceptable.

# Problem 21

You are driving along the highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant decceleration is required to stop your car in 242 ft?

# Problem 22

Let $f$ and $g$ be continuous and differentiable functions satisfying the given conditions for some real number $B$:

I. $\int_1^3 f(x+2)\,dx = 3B$

II. The average value of $f$ in the interval $[1,3]$ is $2B$

III. $\int_{-4}^{x} g(t)\,dt = f(x) + 3x$

IV. $g(x) = 4B + f'(x)$

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**(a)** Find $\int_1^5 f(x)\,dx$ in terms of $B$.

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**(a)** Find $\int_1^5 f(x)\,dx$ in terms of $B$.

**(b)** Find $B$.

# Problem 23

Solve the differential equation $dP/dt = kP^2$ for constant $k$, with initial condition $P(0) = P_0$. Prove that the graph of the solution has a vertical asymptote at a positive value of $t$. What is that value of $t$? (This value is called the *catastrophic solution*.)

# Problem 24

A particle is moving along the path of the curve determined by the parametric equations $x = e^{2t} + 1$ and $y = \ln(e^{4t} + 2e^{2t} + 1)$, where $t > 0$.

**(a)** Find $dy/dx$ in terms of $t$, then find the equation of the tangent line at time $t = \frac{1}{2}$.

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**(c)** Sketch the path of the curve and indicate the direction of motion.

**(d)** Write the set of parametric equations without the parameter $t$.

# Problem 25

Find all real-valued continuously differentiable functions $f$ on the real line such that for all $x$,

$$(f(x))^2 = \int_0^x \left[(f(t))^2 + (f'(t))^2\right] \, dt + 1990.$$

# Problem 26

During your teacher's days as a student last century, he often studied calculus in a dim unheated room with only one candle for light and heat. One particular day in mid-winter, after walking 10 miles (uphill both ways!) through knee-deep snow to attend class, he returned home too tired to study. After lighting the solitary candle on his desk, he walked directly away cursing his woeful situation. The temperture (in degrees Fahrenheit) and illumination (in percentage of candle-power) decreased as his distance (in feet) from his candle increased. In fact, he kept a record of this and in Table 3 is that information, just in case you may not believe the preceding sad tale!

# Table 3

| Distance (feet) | Temperature (°F) | Illumination (% candle-power) |
|---|---|---|
| 0 | 55.0 | 100 |
| 1 | 54.5 | 88 |
| 2 | 53.5 | 77 |
| 3 | 52.0 | 68 |
| 4 | 50.0 | 60 |
| 5 | 47.0 | 56 |
| 6 | 43.5 | 53 |

# Problem 26 continued

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

# Problem 26 continued

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

**(a)** What is the average rate at which the temperature is changing when the illumination drops from 77% to 56%?

# Problem 26 continued

Assume that I get cold when the temperature is below $40^\circ$F and it is dark when the illumination is at most $50\%$ of one candle-power.

**(a)** What is the average rate at which the temperature is changing when the illumination drops from $77\%$ to $56\%$?

**(b)** I can still read my old unlit analog watch when the illumination is $64\%$. Can I still read my watch when I am 3.5 feet from the candle? Explain.

# Problem 26 continued

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

**(a)** What is the average rate at which the temperature is changing when the illumination drops from 77% to 56%?

**(b)** I can still read my old unlit analog watch when the illumination is 64%. Can I still read my watch when I am 3.5 feet from the candle? Explain.

**(c)** Suppose that at 6 feet the instantaneous rate of change of the temperature is −4.5°F per foot and the instantaneous rate of change of the illumination is −3% candle-power per foot. Estimate the temperature and the illumination at 7 feet.

# Problem 26 continued

Assume that I get cold when the temperature is below 40°F and it is dark when the illumination is at most 50% of one candle-power.

**(a)** What is the average rate at which the temperature is changing when the illumination drops from 77% to 56%?

**(b)** I can still read my old unlit analog watch when the illumination is 64%. Can I still read my watch when I am 3.5 feet from the candle? Explain.

**(c)** Suppose that at 6 feet the instantaneous rate of change of the temperature is $-4.5$°F per foot and the instantaneous rate of change of the illumination is $-3\%$ candle-power per foot. Estimate the temperature and the illumination at 7 feet.

**(d)** Am I in the dark before I am cold or am I cold before I am in the dark? Explain.

# Problem 27

Compute $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}$.

# Problem 28

Jay is a waiter at a fine-dining restaurant with 100 tables. During his first month he waited on 20 tables every night, and collected an average tip of \$15 from each table. He started to work more tables, and noticed that for every extra table he took on in a night, his average tip would go down 25 cents per table. He figures that he is physically capable of waiting on up to 30 tables in a night. If Jay wants to maximize his tip money, how many more tables should he wait on?