Implementing Changes in the AP Calculus BC Curriculum

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Outline

New Curriculum Framework

Changes to BC

Implementing the Changes
- Limit Comparison Test
- Absolute and Conditional Convergence
- Alternating Series Error Bound

Conclusion

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New Framework Replaces Topic Outline

Big Idea

- Enduring Understanding
    - Learning Objectives
        - Essential Knowledge

The specifics of the content are found in the "Essential Knowledge" sections.

Interesting Emphasis: Limits

EK 1.1A3: A limit might not exist for some functions at particular values of x. Some ways that the limit might not exist are if the function is unbounded, if the function is oscillating near this value, or if the limit from the left does not equal the limit from the right.

EXAMPLES OF LIMITS THAT DO NOT EXIST:

$$\lim_{x\to 0} \frac{1}{x^2} = \infty \qquad \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \textit{ does not exist}$$

$$\lim_{x\to 0} \frac{|x|}{x} \textit{ does not exist} \qquad \lim_{x\to 0} \frac{1}{x} \textit{ does not exist}$$

Interesting Emphasis: Integrals

EK 3.2C3: The definition of the definite integral may be extended to functions with removable or jump discontinuities.

EK 3.3B4: Many functions do not have closed form antiderivatives.

Interesting Emphasis: Riemann Sums

EK 3.2A2: The definite integral of a continuous function f over the interval $[a, b]$, denoted by $\int_a^b f(x)\ dx$ is the limit of Riemann sums as the widths of the subintervals approach 0. That is, $\int_a^b f(x)\ dx = \lim\limits_{\max \Delta x_i \to 0} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$ where x_i^* is a value in the ith subinterval, Δx_i is the width of the ith subinterval, n is the number of subintervals, and $\max \Delta x_i$ is the width of the largest subinterval. Another form of the definition is $\int_a^b f(x)\ dx = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = (b-a)/n$ and x_i^* is a value in the ith subinterval.

Interesting Emphasis: Riemann Sums Problem

8. (Page 31) Which of the following limits is equal to $\int_3^4 x^4\,dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{k}{n}\right)^4 \frac{2}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(3+\frac{2k}{n}\right)^4 \frac{2}{n}$

Interesting Emphasis: Differential Equations

EK 3.5A2: Some differential equations can be solved by separation of variables.

EK 3.5A3: Solutions to differential equations may be subject to domain restrictions.

EK 2.3E1: Solutions to differential equations are functions or families of functions.

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Topics Added to Infinite Series

- The limit comparison test
- Absolute and conditional convergence
- The alternating series error bound

Topics Added to Infinite Series: Convergence

EK 4.1A4: A series may be absolutely convergent, conditionally convergent, or divergent.

EK 4.1A5: If a series converges absolutely, then it converges.

EK 4.1A6: In addition to examining the limit of the sequence of partial sums of the series, methods for determining whether a series of numbers converges or diverges are the nth term test, the comparison test, the limit comparison test, the integral test, the ratio test, and the alternating series test.

Topics Added to Infinite Series: Estimate Sums

EK 4.1B2: If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series.

EK 4.1B3: If a series converges absolutely, then any series obtained from it by regrouping or rearranging the terms has the same value.

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Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n \geq 0$ and $b_n > 0$. Let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then we have the following three cases.

(A) If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

(B) If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ also converges.

(C) If $L = 0$ and $\sum a_n$ diverges, then $\sum b_n$ also diverges.

Limit Comparison Test: a Problem

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}.$$

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Direct Comparison Test doesn't work!

Limit Comparison Test: the Solution

Determine the convergence or divergence of

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Limit Comparison Test: the Solution

Determine the convergence or divergence of

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}.$$

We use $\sum \frac{1}{3^n}$ which converges by the Geometric Series Test.

$$L = \lim_{n\to\infty} \frac{1/3^n}{1/(3^n - n)} = \lim_{n\to\infty} \frac{3^n - n}{3^n} = 1$$

Hence, $\sum \frac{1}{3^n - n}$ converges by the Limit Comparison Test.

Limit Comparison Test: Which Comes First?

Whether you compute

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \text{or} \quad \lim_{n\to\infty} \frac{b_n}{a_n}$$

does not really matter: If L is positive and finite, so is $1/L$.

Limit Comparison Test: Another Problem

Is the infinite series

$$\sum_{n=1}^{\infty} \frac{3 + n \ln n}{n^2 + 7}$$

convergent or divergent?

Limit Comparison Test: Another Solution

For large n, we expect the expression to behave like $n\ln n/n^2 = \ln n/n$. So we compare to $\sum \ln n/n$ which diverges. Then

$$L = \lim_{n\to\infty} \frac{\frac{3+n\ln n}{n^2+7}}{\frac{\ln n}{n}} = \lim_{n\to\infty} \frac{3n+n^2\ln n}{n^2\ln n+7\ln n} = 1.$$

Thus $\sum \frac{3+n\ln n}{n^2+7}$ diverges.

Absolute and Conditional Convergence

From an infinite series $\sum a_n$ we may form a series of its absolute values $\sum |a_n|$. If $\sum |a_n|$ is convergent then we say that $\sum a_n$ is *absolutely convergent*, or that it *converges absolutely*.

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Theorem (Absolute Convergence Theorem)

Every absolutely convergent series is also convergent. In other words, if $\sum |a_n|$ converges, so does $\sum a_n$.

Absolute and Conditional Convergence: a Problem

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Since $1/\sqrt{n-1}$ is positive, decreasing, and goes to zero, the series converges by the Alternating Series Test. Since $1/\sqrt{n-1} > 1/\sqrt{n}$, and $\sum 1/\sqrt{n}$ is a divergent p-series, the series $\sum 1/\sqrt{n-1}$ diverges by the Direct Comparison Test. Therefore, the series is conditionally convergent.

Absolute and Conditional Convergence: Another Problem

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The series of absolute values is simply $5 \sum 1/n^3$, which is a convergent p-series. Therefore, the series is absolutely convergent by the Absolute Convergence Theorem.

Absolute and Conditional Convergence: Yet Another Problem

Determine the absolute convergence, conditional convergence, or divergence of

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Note that $(n^2+1)/(2n^2+3)$ is positive and decreasing, but the terms approach $\frac{1}{2}$, not zero. Therefore the series diverges by the Alternating Series Test.

Absolute and Conditional Convergence: Rearrangement Theorem

Theorem (Rearrangement Theorem)

Let $\sum a_n$ be an absolutely convergent series, and let $\sum r_n$ be the sum of a rearrangement of the terms of $\sum a_n$. If $\sum a_n$ converges to L, $\sum r_n$ converges to L.

Absolute and Conditional Convergence: Why Rearrangement Matters

The series $\sum(-1)^{n-1}\frac{1}{n}$ is conditionally convergent. Since this is the Taylor series for $f(x) = \ln(x+1)$ at $x = 2$, we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots = \ln 2.$$

But! Take two odd terms followed by an even term...

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2.$$

Absolute and Conditional Convergence: Conditional Rearrangement Fails

Note that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

and

$$\frac{1}{2}\ln 2 = \qquad \frac{1}{2} \qquad - \frac{1}{4} \qquad + \frac{1}{6} \qquad - \frac{1}{8} + \cdots .$$

Both series converge, so we may add them together.

$$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots .$$

Alternating Series Error Bound

- Part of the old course description; "Alternating series with error bound"
- Now called out specifically: "**EK 4.1B2**: If an alternating series converges by the alternating series test, then the alternating series error bound can be used to estimate how close a partial sum is to the value of the infinite series."

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- Ben Hedrick: "...we're calling out some things by name, which means that we can also test them by name. We have in the past asked for error estimates, and it was implied that students would be using the Alternating Series Error Bound. Now we can ask for it specifically (and more clearly)."

Alternating Series Error Bound Theorem

Theorem (Alternating Series Error Bound)

Let $\sum(-1)^n a_n$ be a convergent alternating series. The error in using the kth partial sum s_k to estimate the sum S is less than the $(k+1)$th term of the series a_{k+1}.

Alternating Series Error Bound: a Problem

Approximate the sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

by using s_{10} and determine the maximum error.

Alternating Series Error Bound: the Solution

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$$s_{10} = \sum_{n=1}^{10} \frac{(-1)^{n+1}}{n^4} = 0.9469925924$$

The error in using this approximation is

$$|S - s_{10}| < a_{11} = \frac{1}{11^4} \approx 0.0000683.$$

Hence, the interval

$$(s_{10} - a_{11}, s_{10} + a_{11}) = (0.9469243, 0.9470608)$$

must contain the sum S.

Alternating Series Error Bound: AP Problem

4. (Page 51) The Taylor series for a function f about $x = 0$ onverges to f for $-1 \leq x \leq 1$. The nth degree Taylor polynomial for f about $x = 0$ is given by $P_n(x) = \sum_{k=1}^{n} (-1)^k \frac{x^k}{k^2+k+1}$. Of the following, which is the smallest number M for which the alternating series error bound guarantees that $|f(1) - P_4(1)| \leq M$?

(A) $\frac{1}{5!} \cdot \frac{1}{31}$

(B) $\frac{1}{4!} \cdot \frac{1}{21}$

(C) $\frac{1}{31}$

(D) $\frac{1}{21}$

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Thanks!

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