Discrete Calculus

(Most of) Calculus Through Euler's Method

Chuck Garner, Ph.D.

Department of Mathematics
Rockdale Magnet School for Science and Technology

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Outline

My Journey

Rates & Predictions

More Connections

Results

What is Euler's Method?

A numerical recursive procedure used to approximate the solution of a differential equation at a point.

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Given a differential equation $\frac{dy}{dx} = f(x, y)$ and initial value $y_0 = y(x_0)$, then the solution y_n to the differential equation at the point $x = x_n$ is approximated by

$$y_{k+1} \approx \frac{dy}{dx}(x_k, y_k)\Delta x + y_k$$

for $k = 0, 1, \dots, n$, where $\Delta x = \frac{x_n - x_0}{n}$.

How Do I Use Euler's Method?

Approximate $y(2)$ given $y(0) = 1$ and $\frac{dy}{dx} = x^2 + 3y$, using a step size of $\Delta x = 1$.

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$$y(1) \approx (0^2 + 3 \cdot 1)(1) + 1 = 4 \qquad \text{new point: } (1, 4)$$

$$y(2) \approx (1^1 + 3 \cdot 4)(1) + 4 = 17 \qquad \text{new point: } (2, 17)$$

Answer: $y(2) \approx 17$.

What's Wrong With This?

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- Confusing
- Comes out of nowhere
- Why use it when we can integrate?

Euler's Tangents

Problem: Write the equation of the line tangent to $y = x^2$ where $x = 3$.

- To write a tangent line, one needs a *slope* and a *point*
- We have $y' = 2x$
- At $(3, 9)$, slope is 6
- Use point-slope form to write

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- Rewrite:

$$y = 6(x - 3) + 9$$

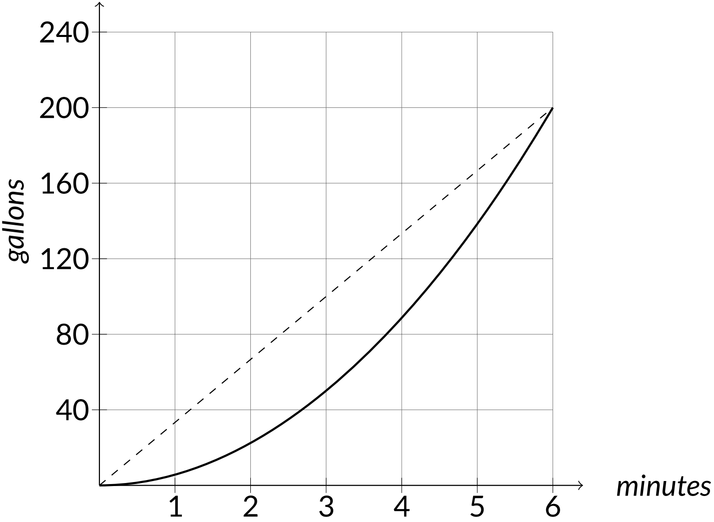
This is the Euler's Method equation!

Euler's Tangents

Tangent: $$y = f'(x_0)(x - x_0) + y_0$$

Euler's Method: $$y_{k+1} = \frac{dy}{dx}(x_k, y_k)\Delta x + y_k$$

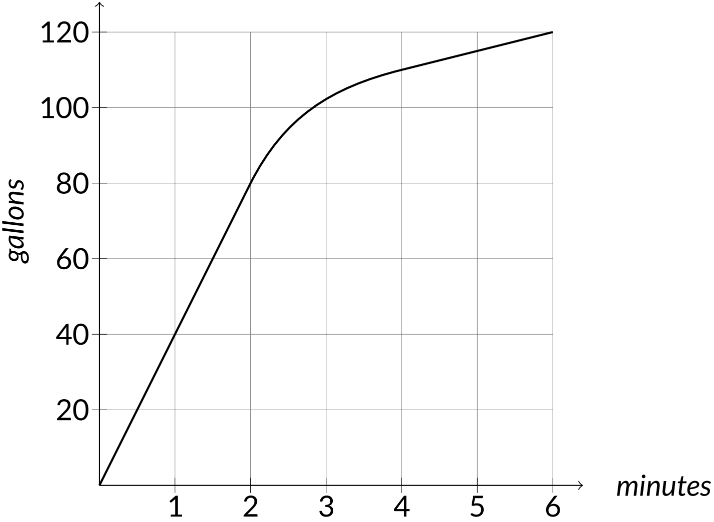
Water Flowing Into a Tank



Describing Water Flow

At time $t = 0$, water begins to flow from a tap into an empty tank at the rate of 40 gallons per minute. This flow rate is held constant for 2 minutes, then the tap is gradually closed until at $t = 4$ the flow rate is 5 gallons per minute. This new rate is held constant for the final 2 minutes, so that at time $t = 6$, the tank contains 120 gallons.

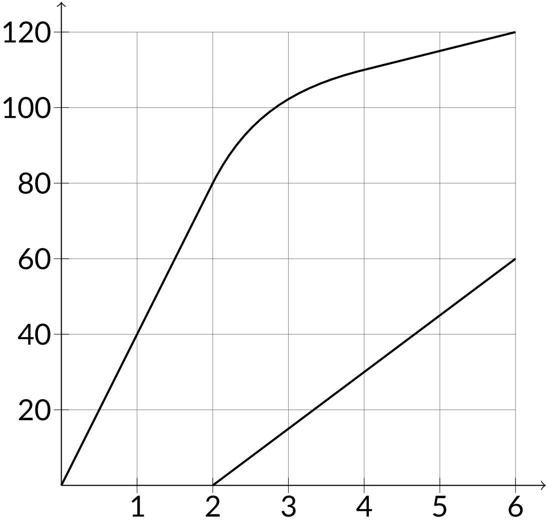
Describing Water Flow



Describing Water Flow

Now suppose that a pump is started at $t = 2$, and, for the next 4 minutes, pumps water out of the tank at a constant rate of 15 gallons per minute. The previous graph you drew now represents the total amount of water flowing into the tank at time t. On the same axes as your volume graph, draw the graph that represents the total amount of water pumped out at time t. How do you now intepret the *total* amount of water at time t?

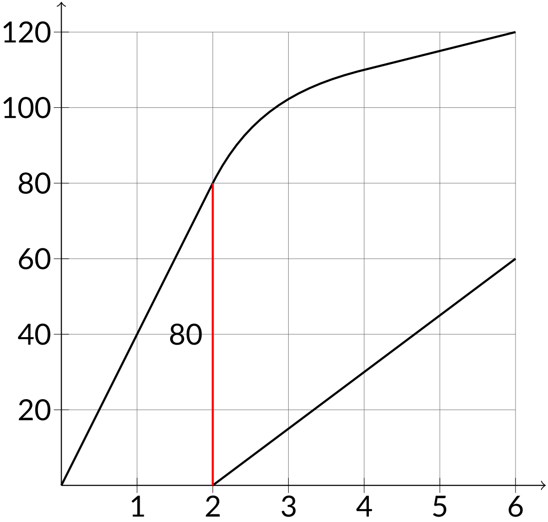
Describing Water Flow



Describing Water Flow

Show how to find the maximum amount of water in the tank.

Describing Water Flow



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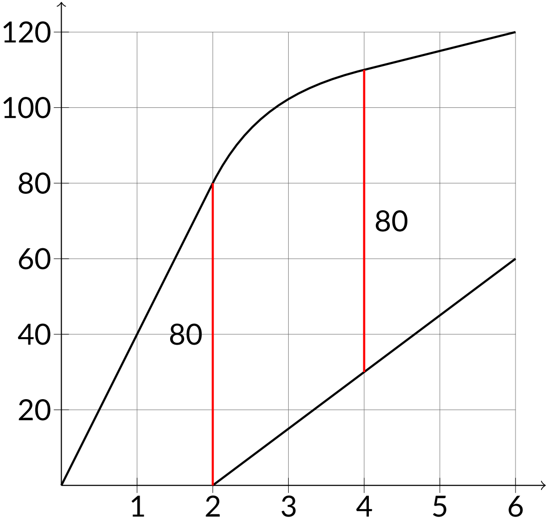

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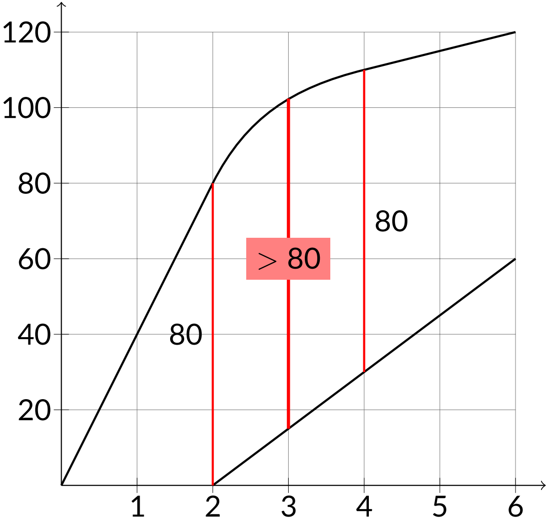

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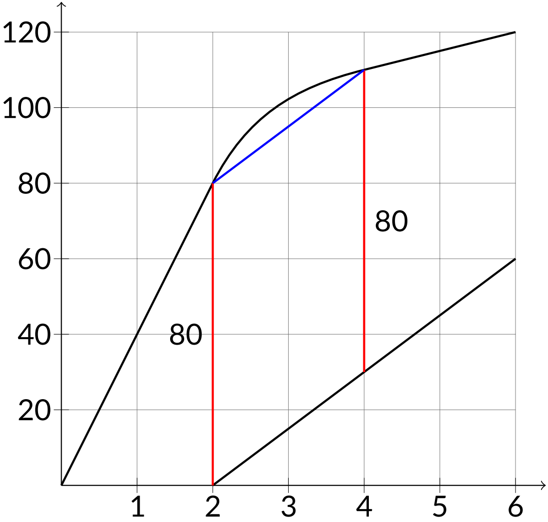

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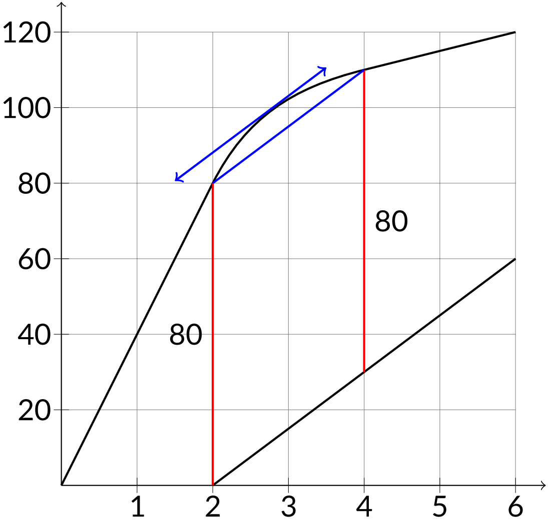

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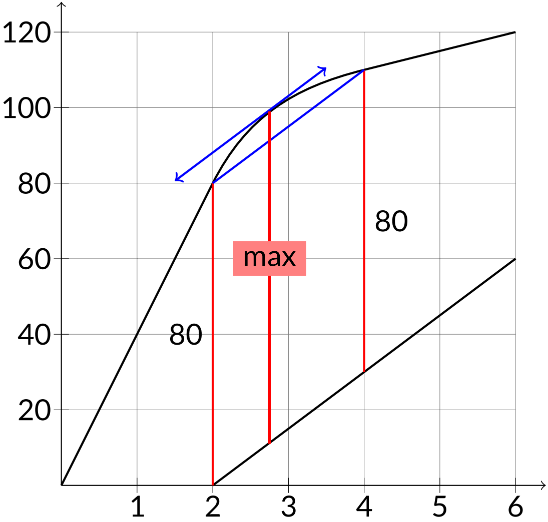

A New Approach

- Investigate rates of change in various applications
- Investigate changing quantities from tables of data
- Let Euler's method arise naturally
- Connect *rate* to *slope* and later to *derivative*

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Rates Everyday

"The population is growing more slowly."
"The plane is landing smoothly."
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"In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting President used the third derivative to advance his case for re-election." —*Hugo Rossi*

A Rate Problem

A tank is being filled at a variable rate. The only thing known is that between 1 and 2 minutes, the average inflow rate is 18 gallons per minute, and the amount of water in the tank at 2 minutes is 42 gallons. What is your best estimate for the amount of water in the tank at 2.5 minutes?

Population Growth

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Malthus' Equation:

$$P' = kP$$

where k is constant and P is the population is a rate equation for population growth

Let's use it to make predictions!

Population Growth

Population of the state of Georgia

Year	Population (millions)
1900	2.216
1910	2.609
1920	2.896
1930	2.909
1940	3.124
1950	3.445
1960	3.943
1970	4.590
1980	5.463
1990	6.478
2000	8.186

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and use table values to approximate P'. Using 1900 to 1910, we have

$$P' \approx \frac{2.609 - 2.216}{191 - 190} = \frac{0.393}{1} = 0.393.$$

Therefore,

$$k = \frac{P'}{P} \approx \frac{0.393}{2.609} = 0.151.$$

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We compute:

$$(8.186 \times 0.124) + 8.186 = 9.201 \text{ million in 2010, and}$$

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$$(9.201 \times 0.124) + 9.201 = 10.342 \text{ million in 2020.}$$

Population Growth

Problem.
Is it possible to predict the population of Georgia in the year 2015? Either explain why it cannot be done, or give a method for doing so.

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At what value of P will the growth rate be the largest?
This is quadratic in P. The graph of $MP - P^2$ is a parabola opening downwards; hence, $P(M - P)$ has a maximum at its vertex. Its vertex occurs when $P = \frac{1}{2}M$.

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To approximate the population when $t = 1$, we use the same method as before with regular population growth.

$$P' = 0.0016(10)\,(500 - 10) = 0.016\,(490) = 7.84,$$

then the population at $t = 1$ is $10 + 7.84 = 17.84$.

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then the population at $t = 1$ is $10 + 7.84 = 17.84$.
We repeat the process to estimate the population at $t = 2$:

$$P' = 0.0016(17.84)(500 - 17.84) = 13.763$$

and the population at 2 hours is then $17.84 + 13.763 = 31.603$.

Velocities

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Given $w = 527$, $b = 470$, $k = 0.08$, and $m = 16.316$, we have

$$v' = \frac{57 - 0.08v}{16.316} = 3.494 - 0.0049v.$$

Euler's Method in This Context

Given a rate equation P' that describes the rate of the quantity P at time t, then successive points are approximated by

$$P(t_{n+1}) \approx P(t_n) + P'(t_n)\Delta t,$$

where (t_0, P_0) is the initial value and Δt is the step size. This approximation technique is called *Euler's method*.

Euler's Method in This Context

Problem. The Mauna Loa Observatory (MLO) in Hawaii has a record of carbon dioxide levels in the atmosphere going back to March 1958. Levels of carbon dioxide are measured in parts per million (PPM). The average rate of change of CO_2 levels is 1.397 PPM/yr.

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(a) Given an initial condition of 315.71 PPM in March 1958, find the constant k so that the CO_2 levels can be modeled by a rate equation of the form $C' = kC$, where C is measured in PPM.

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(c) The Intergovernmental Panel on Climate Change (IPCC) projects that CO_2 levels could reach 450-550 PPM by 2050, possibly resulting in higher temperatures and rising sea levels. Does your mathematical model confirm this? Explain.

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From Euler to Riemann

$$
\begin{aligned}
y(x_1) &= y'(x_0)\Delta x + y(x_0), \\
y(x_2) &= y'(x_1)\Delta x + y(x_1), \\
y(x_3) &= y'(x_2)\Delta x + y(x_2), \\
&\vdots \\
y(x_n) &= y(x_{n-1}) + y'(x_{n-1})\Delta x.
\end{aligned}
$$

From Euler to Riemann

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y(x_n) - y(x_{n-1}) &= y'(x_{n-1})\Delta x.
\end{aligned}
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From Euler to Riemann

$$y(x_n) - y(x_0) = \left[y'(x_0) + y'(x_1) + \cdots + y'(x_{n-1})\right] \Delta x$$

$$y(x_n) - y(x_0) = \Delta x \sum_{k=0}^{n-1} y'(x_k).$$

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A discrete version of the Fundamental Theorem of Calculus.

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Teaching

- Begin the year with “rate equations”
- 2-3 weeks: Tanks, Populations, Barrels, Euler’s Method, Slope Fields
- Derivatives from a rate perspective
- Integrals as Riemann sums
- Differential equations is the reason we do calculus

Learning

Students...

- no longer wonder what a derivative means
- understand that approximate methods are adequate in certain situations
- understand that exact methods are preferrable (if possible)
- realize that the derivative is quick, exact, and easier to use than Euler's method
- have insight into the FTC before integrals are introduced

Thanks!

E-Mail: cgarner@rockdale.k12.ga.us

Website: http://web.me.com/drcgarner

The Georgia Association of Advanced Placement Math Teachers
GA^{2}PMT
http://gaapmt.wikispaces.com