My Favorite Algebra Problems

A Personal Selection of Algebra Challenges for Student Enrichment or Math Teams

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October 18, 2007 / Georgia Math Conference - Rock Eagle

Outline

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Roots: Basic Facts

Viète's Relations. Given $x^3 + ax^2 + bx + c = 0$ where r, s, and t are the roots, then

$$\begin{aligned} r + s + t &= -a \\ rs + rt + st &= b \\ rst &= -c \end{aligned}$$

Roots: Basic Facts

Easily generalized for $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ with roots $r_1, r_2, \ldots, r_n$.

$$\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}$$

$$\sum_{i,j=1,i\neq j}^{n} r_ir_j = \frac{a_{n-2}}{a_n}$$

$$\prod_{i=1}^{n} r_i = (-1)^n\frac{a_0}{a_n}$$

$$\sum_{i=1}^{n} r_i^2 = \frac{a_{n-1}^2 - 2a_{n-2}a_n}{a_n^2}$$

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Clearly, $r + s = 21$ and $rs = 10$. Since

$$(r+s)^2 = r^2 + 2rs + s^2$$

we have

$$21^2 = r^2 + 2(10) + s^2,$$

or

$$r^2 + s^2 = 441 - 20 = 421.$$

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Polynomials: Basic Facts

The Factor Theorem. If $P(a) = 0$, then $x - a$ is a factor of $P(x)$.
The Remainder Theorem. If a polynomial $P(x)$ is divided by $x - a$, then the remainder is the value of $P(a)$.

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The Sum of the Coefficients. The value $P(1)$ is the sum of the coefficients of $P(x)$.
Use the roots!

Polynomials: Problem 1

Steve picks a fourth degree polynomial p with nonnegative integer coefficients and challenges Jack to discover the five coefficients. Steve lets Jack pick only two values of x to help him discover the coefficients. Jack picks $x = 1$ and $x = 10$. Steve tells him $p(1) = 9$ and $p(10) = 32,013$. Now Jack knows precisely what the polynomial's coefficients are, and as proof, Jack tells Steve the value of $p(3)$. What is the value of $p(3)$?
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$p(x) = 3x^4 + 2x^3 + x + 3$. Hence, $p(3) = 303$.

Polynomials: Problem 2

The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?
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Because $P(x)$ has three roots, and if $Q_1(x)$ and $Q_2(x)$ are both factors of P, then they must have a common root r. Then $Q_1(r) + Q_2(r) = 0$, and

$$mQ_1(r) + nQ_2(r) = 0$$

for any values of m and n.

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$$Q_1(r) = \frac{k^2}{25} - (k - 29)\frac{k}{5} - k = 0$$

which is equivalent to $4k^2 - 120k = 0$. The larger root is $k = 30$.

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[$k = 30$ implies $Q_1(x) = (x+6)(x-5)$, $Q_2(x) = (2x+5)(x+6)$, and $P(x) = (x+6)(x-5)(2x+5)$.]

Polynomials: Problem 3

If $f(x) = 3x^2 - 2x + 5$ and $f(g(x)) = 12x^4 + 56x^2 + 70$, then compute the product of all possible values of the sum of the coefficients of $g(x)$.
[ARML Practice, 2007]

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The sum of the coefficients of $g(x)$ is $g(1)$.
Since $f(g(x)) = 3(g(x))^2 - 2g(x) + 5$, we have

$$f(g(1)) = 3(g(1))^2 - 2g(1) + 5 = 12 + 56 + 70,$$

or $3(g(1))^2 - 2g(1) - 133 = 0$.

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or $3(g(1))^2 - 2g(1) - 133 = 0$. We factor to obtain

$$[3g(1) + 19][g(1) - 7] = 0$$

so that $g(1) = 7$ or $g(1) = -19/3$. The product is $-133/3$.

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Example. The sum

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Then we see that the "middle" terms cancel:

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{k+1}.$$

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Rewrite the given expression.

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We write

$$\frac{k}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}.$$

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Then the sum becomes

$$\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \cdots + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1 - \frac{1}{(n+1)!}$$

Telescoping Sums: Problem 2

A sequence is defined as follows: $a_1 = a_2 = a_3 = 1$, and, for all positive integers n,

$$a_{n+3} = a_{n+2} + a_{n+1} + a_n.$$

Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when

$$\sum_{k=1}^{28} a_k$$

is divided by 1000.
[AIME II, 2006]

Telescoping Sums: Problem 2

Because $a_{k+3} - a_{k+2} = a_{k+1} + a_k$ for all positive integers k, we have

$$\sum_{k=1}^{n}(a_{k+3} - a_{k+2}) = \sum_{k=1}^{n}(a_{k+1} + a_k)$$

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Let $S_n = \sum_{k=1}^{n} a_k$. Notice that $\sum_{k=1}^{n}(a_{k+3} - a_{k+2})$ is telescoping, and becomes simply $a_{n+3} - a_3$.

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Thus $a_{n+3} - a_3 = S_n - a_1 + a_{n+1} + S_n$, so that

$$S_n = \frac{1}{2}(a_{n+3} - a_{n+1}) = \frac{1}{2}(a_{n+2} + a_n).$$

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$$S_n = \frac{1}{2}(a_{n+3} - a_{n+1}) = \frac{1}{2}(a_{n+2} + a_n).$$

In particular, $S_{28} = \frac{1}{2}(a_{30} + a_{28}) = \frac{1}{2}(20603361 + 6090307) = 13346834$ so the remainder when divided by 1000 is 834.

Outline

Arithmetic-Geometric Sums: Defintions

A sum such as the following

$$\frac{3}{4}+\frac{7}{16}+\frac{11}{64}+\cdots+\frac{3+4k}{4^{k-1}}+\cdots$$

has a numerator that is arithmetic and a denominator that is geometric.

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To evaluate, we set the sum equal to S and compute $S - \frac{1}{4}S$.
This sum is either *geometric* or includes the original series.

Arithmetic-Geometric Sums: Problem 1

Evaluate $\sum_{n=1}^{\infty} \frac{F_n}{10^{n+1}}$, where F_n is the nth Fibonacci number, i.e.,
$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3$, etc.
[GCTM State Tournament, 2005]

Arithmetic-Geometric Sums: Problem 1

Call the sum S. Then

$$S = \frac{1}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{3}{10^5} + \cdots$$

and

$$\frac{1}{10}S = \frac{1}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \frac{3}{10^6} + \cdots$$

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Then

$$S - \frac{1}{10}S = \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \cdots = \frac{1}{100} + \frac{1}{100}S,$$

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or

$$\frac{9}{10}S = \frac{1}{100}(S+1),$$

which implies $90S = S + 1$ whose solution is $S = \dfrac{1}{89}$.

Two Sums and a Product

Let a and r be positive real numbers. If $\sum_{n=1}^{10} ar^n = 18$ and $\sum_{n=1}^{10} \frac{1}{ar^n} = 6$, then find $\prod_{n=1}^{10} ar^n$.

[ARML Practice, 2007]

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$$6 = \frac{1}{ar} + \frac{1}{ar^2} + \cdots + \frac{1}{ar^{10}} = \frac{r^9 + \cdots + r + 1}{ar^{10}} = \frac{18/(ar)}{ar^{10}} = \frac{18}{a^2r^{11}}$$

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Hence, $a^2r^{11} = 18/6 = 3$.

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Hence, $a^2r^{11} = 18/6 = 3$. Thus,

$$a^{10}r^{55} = (a^2r^{11})^5 = 3^5 = 243.$$

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A Number and Its Reciprocal

If $x + \dfrac{1}{x} = 3$, then find $x^2 + \dfrac{1}{x^2}$.

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If $x + \dfrac{1}{x} = 3$, then find $x^2 + \dfrac{1}{x^2}$.

Squaring the given equation, we have

$$x^2 + 2 + \frac{1}{x^2} = 9,$$

A Number and Its Reciprocal

If $x + \frac{1}{x} = 3$, then find $x^2 + \frac{1}{x^2}$.

Squaring the given equation, we have

$$x^2 + 2 + \frac{1}{x^2} = 9,$$

so then

$$x^2 + \frac{1}{x^2} = 7.$$

Fun with Coefficients

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Let $x = 1$ to get $(1)^{75}(1)^{125} = 1$.

The Law of Cosines

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[GCTM State Tournament 2006]

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The largest angle is opposite the largest side; the largest side is $\sqrt{x^2+xy+y^2}$. By the Law of Cosines, we have

$$\begin{aligned}
\left(\sqrt{x^2+xy+y^2}\right)^2 &= x^2+y^2-2xy\cos\theta \\
x^2+xy+y^2 &= x^2+y^2-2xy\cos\theta \\
xy &= -2xy\cos\theta \\
-\tfrac{1}{2} &= \cos\theta \\
\theta &= 120^\circ
\end{aligned}$$

A System of Equations

Consider positive integers A, B, and C such that $A^2 + B - C = 100$ and $A + B^2 - C = 124$. Find the value of C.
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Nested Radicals

Evaluate

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Let $x = \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}}$. Then upon squaring both sides, we get

$$\begin{aligned} x^2 &= 6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}} \\ x^2 &= 6+x \\ x^2-x-6 &= 0 \\ x &= 3 \end{aligned}$$

Outline

Summary of Basic Techniques

Roots Use Viète's Relations – only find the roots as a last resort

Polynomials Factors, remainders, $f(1)$

Telescoping Series Rewrite the expression

Arithmetic-Geometric Turn it into a telescoping sum