More Favorite Algebra Problems

Dr. Chuck Garner

October 18, 2007

1 The Problems

1. Given that a, b, and c are roots of $x^3 - 3x^2 + mx + 24$ and that $-a$ and $-b$ are roots of $x^2 + nx - 6$, then compute the value of n. *(GCTM State Tournament, 2006)*

2. Given that a, b, and c are the nonzero real roots of $f(x) = x^3 + ax^2 + bx + c$ and that the sum of the squares of the roots is $a - 2b$, then compute the value of $f(2)$. *(GCTM State Tournament, 2007)*

3. The function $y = f(x)$ is a function such that $f(f(x)) = 10x - 2007$ for all real numbers x. An integer n satisfies the equation $f(n) = 10n - 2007$. What is this value of n? *(GCTM State Tournament, 2007)*

4. Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers. Suppose $P(1) = 827$, $P(2) = 1654$, and $P(3) = 2481$. Find the value of $\frac{1}{4}[P(5) + P(-1)]$. *(Georgia ARML Practice, 2007)*

5. Evaluate $\displaystyle\sum_{k=1}^{99} \frac{k+1}{(k-1)! + k! + (k+1)!}$.

6. The remainder of $x^5 - 8x^3 + 7x + 1$ when divided by $x + a$ is 1. Find the sum of all possible values of a. *(GCTM State Tournament, 2005)*

7. If r, s, and t are roots to the equation $x^3 + mx + n = 0$ such that $r = t - 4$, $s = r - 1$, and $t = s + 5$, then find the values of m and n. *(GCTM State Tournament, 2005)*

8. Given that n is a positive integer, determine the value of n that satisfies the equation
$$\frac{n^3-3}{n^3} + \frac{n^3-4}{n^3} + \frac{n^3-5}{n^3} + \frac{n^3-6}{n^3} + \cdots + \frac{5}{n^3} + \frac{4}{n^3} + \frac{3}{n^3} = 169.$$
(GCTM State Tournament, 2005)

9. Evaluate $\displaystyle\sum_{k=1}^{n} \frac{2}{k(k+2)}$ in terms of n. *(GCTM State Tournament, 2005)*

10. Let A be the sum of the coefficients in the expansion of $(2007x - 1)^{2007}$. What is the units digit of A? *(Rockdale Tournament, 2007)*

11. Find the largest real solution to $\sqrt{3-x\sqrt{3-x\sqrt{3-x\sqrt{3-\dots}}}} = x+5$. *(GCTM State Tournament, 2007)*

12. When $(1-2x)^3(1+kx)^2$ is expanded, two values of k give the coefficient of x^2 as 30. The sum of these two values of k is *(GCTM State Tournament, 2007)*

13. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{F_n}{5^n}$ where F_n are the terms in the Fibonacci sequence (i.e., $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$). *(GCTM State Tournament, 2007)*

14. The polynomial with rational coefficients $f(x) = x^4 + bx^3 + cx^2 + dx + e$ has roots $\sqrt{3}$ and $1-5i$. Find $f(1)$. *(GCTM State Tournament ciphering, 2005)*

15. If $f(x) = x^2 + 2x + 2$ and $g(f(x)) = 2x^2 + 4x + 1$, then find $g(7)$. *(GCTM State Tournament ciphering, 2005)*

16. Let k be a real number such that $x^2 + x\sqrt{20} - k = 0$. If k is chosen from the interval $[-7,-2]$, then what is the probability that the roots of the equation are real? *(GCTM State Tournament ciphering, 2005)*

17. The sum of the roots of $f(x) = 5x^3 - px^2 - 3x + q$ is 2. If $f(1) = 3$, then find the value of q. *(Rockdale Tournament, 2007)*

18. Find all numbers x such that $(x+i)^4 = x^4 - 6x^2 + 1$, where $i = \sqrt{-1}$. *(GCTM State Tournament ciphering, 2006)*

19. Evaluate $\displaystyle\sum_{k=0}^{20} k[k(k)-k](k-2)!$ *(Rockdale Tournament, 2006)*

20. Compute: $\dfrac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}$ *(AIME, 1987)*

21. Evaluate $\displaystyle\sum_{k=1}^{100} \frac{4k}{4k^4+1}$.

22. What is the sum of the coefficients in the expansion of $(x+y+z)^8$? *(Rockdale Tournament, 2006)*

23. Evaluate $\sqrt{30+\sqrt{30+\sqrt{30+\cdots}}}$ *(Rockdale Tournament, 2006)*

24. All three roots of the equation $x^3 - 18x^2 + 107x - 210 = 0$ are positive integers. What is the sum of the cubes of those roots? *(Rockdale Tournament, 2006)*

25. For triangle ABC, $AB = c$, $AC = b$, and $BC = a$. In ΔABC,

$$2a^2 + b^2 + 4c^2 = 2ab + 4ac.$$

Compute the numerical value of $\cos B$. *(ARML, 1983)*

2 The Solutions

1. Since $(-a)(-b) = ab = -6$ and $abc = -24$, we get that $c = 4$. Then, since $a+b+c = 3$, we see that $a+b = -1$. The value of n is given by $n = -[(-a)+(-b)] = a+b = -1$.

2. We have $a+b+c = -a$, and upon squaring this equation, we have $a^2+b^2+c^2+2(ab+ac+bc) = a^2$. Since $ab+ac+bc = b$ and $a^2+b^2+c^2 = a-2b$, we get $a-2b+2b = a^2$, or $a = a^2$. Thus, $a = 1$. The product of the roots is $abc = -c$, or $ab = -1$. Since $a = 1$, we have $b = -1$. Finally, we use our first equation again to get $c = -1$. Hence the function is $f(x) = x^3+x^2-x-1$ and then $f(2) = 9$.

3. We have $f(f(n)) = 10n-2007 = f(n)$, so that $f(n) = n$. Hence, we solve $10n-2007 = n$ to get $n = 223$.

4. Let $Q(x) = P(x) - 827x$. Then $Q(x)$ is of degree 4. Since

$$Q(1) = P(1)-827 = 0, \quad Q(2) = P(2)-1654 = 0, \quad Q(3) = P(3)-2481 = 0,$$

we have $Q(x) = (x-1)(x-2)(x-3)(x-r)$ for some real r. Thus,

$$\frac{1}{4}[P(5)+P(-1)] = \frac{1}{4}[(Q(5)+4135)+(Q(-1)-827)] = \frac{1}{4}[Q(5)+Q(-1)]+827$$

which gives

$$\frac{1}{4}[(4)(3)(2)(5-r)+(-2)(-3)(-4)(-1-r)]+827.$$

Hence, we have $\frac{1}{4}[P(5)+P(-1)] = 6[6]+827 = 863$.

5. We rewrite the expression.

$$\begin{aligned}\frac{k+1}{(k-1)!+k!+(k+1)!} &= \frac{k+1}{(k-1)![1+k+k(k+1)]} \\ &= \frac{k+1}{(k-1)!(k+1)^2} \\ &= \frac{1}{(k-1)!(k+1)} = \frac{k}{(k+1)!}\end{aligned}$$

By *Telescoping Sums: Problem 1* in the presentation, we have

$$\sum_{k=1}^{99} \frac{k+1}{(k-1)!+k!+(k+1)!} = \sum_{k=1}^{99} \frac{k}{(k+1)!} = 1 - \frac{1}{100!}.$$

6. To leave a remainder of 1, $x^5 - 8x^3 + 7x$ must equal zero. Since there is no x^4 term, the sum of the solutions is zero.

7. The relations given between the roots in terms of r are $s = r-1$ and $t = r+4$. Since there is no x^2 term, the sum of the roots is zero; thus, $r+s+t = 3r+3 = 0$ which implies $r = -1$. Hence, since the constant term is the negative product of the roots, we have $n = -rst = -(-1\cdot -2\cdot 3) = -6$. Finally, $m = rs+rt+st = (-1)(-2)+(-1)(3)+(-2)(3) = 2-3-6 = -7$.

8. Mutiplying both sides by n^3, we see that the left side is simply the sum of the integers from 3 to n^3-3, resulting in the equation $\frac{1}{2}(n^3-3)(n^3-3+1)-3 = 169n^3$. This simplifies to the equation $n^6 = 343n^3$; thus, $n = 7$.

9. Note that $\dfrac{2}{k(k+2)} = \dfrac{1}{k} - \dfrac{1}{k+2}$. Then the sum telescopes; the only terms left in the sum are

$$1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3n^2+5n}{2n^2+6n+4}.$$

10. Let $x = 1$. Then the sum of coefficients is 2006^{2007}. Since any power of 6 ends in 6, the units digit is 6.

11. Square both sides and we have $3 - x(x+5) = (x+5)^2$. Setting this equal to zero gives $2x^2 + 15x + 22 = 0$ whose solutions are $x = -\frac{11}{2}$ and $x = -2$. Note that -2 is the only solution since $-\frac{11}{2}$ makes the right-hand side of the original equation negative.

12. We expand each binomial to get

$$(1 - 6x + 12x^2 - 8x^3)(1 + 2kx + k^2x^2).$$

From here, it is easily seen that the coefficient of the x^2 term will be $k^2 - 12k + 12$. The sum of the two solutions is 12.

13. Call the sum S. Then we have

$$S = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{5}{5^4} + \frac{8}{5^5} + \cdots$$

and

$$5S = 1 + \frac{2}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \frac{8}{5^4} + \cdots.$$

Hence,

$$\begin{aligned}
5S - S &= 1 + \frac{1}{5} + \frac{1}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \frac{5}{5^5} + \cdots \\
4S &= 1 + \frac{1}{5} + \frac{1}{5}\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{5}{5^4} + \cdots\right) \\
4S &= \frac{6}{5} + \frac{1}{5}S \\
\frac{19}{5}S &= \frac{6}{5} \\
S &= \frac{6}{19}
\end{aligned}$$

14. The other roots are $-\sqrt{3}$ and $1 + 5i$; thus, $f(x) = (x^2 - 3)(x - 1 - 5i)(x - 1 + 5i)$. Then $f(1) = (1-3)(-5i)(5i) = -2 \cdot 25 = -50$.

15. Clearly, setting $f(x) = 7$ gives the correct value of x. That value is $-1 + \sqrt{6}$ (or $-1 - \sqrt{6}$—either one works); plugging this into $g(f(x))$ results in 11.

16. The roots are real if the discriminant $20 + 4k > 0$, or $k > -5$. The probability is then the length of the favorable interval $[-5, -2]$ divided by the length of the entire interval $[-7, -2]$: $\frac{3}{5}$.

17. The sum of the roots is given by $\frac{p}{5}$ which we are told is 2; thus, $p = 10$. Since $f(1) = 3$, we have

$$5(1^3) - 10(1^2) - 3(1) + q = 3$$

which gives $q = 11$.

18. Expanding $(x+i)^4$ we have $x^4+4ix^3-6x^2-4ix+1$. For this to equal x^4-6x^2+1, we must have that $4ix^3=4ix$. This implies $x^2=1$ or $x=0$; hence, $x=-1,0,1$.

19. First, we factor the expression to get $k\cdot k(k-1)(k-2)!=k\cdot k!$. Then we can write

$$k\cdot k! = (k+1-1)k! = (k+1)k! - k! = (k+1)! - k!.$$

Hence, the sum

$$\sum_{k=0}^{20}[(k+1)!-k!]$$

telescopes, and becomes simply $21!-1$.

20. Notice that $324=4(3^4)$. This is the key to the whole problem, because now each term is of the form x^4+4y^4, and this factors:

$$x^4+4y^4 = x^4+4x^2y^2+4y^4-4x^2y^2 = (x^2+2y^2)^2-(2xy)^2 = (x^2+2xy+2y^2)(x^2-2xy+2y^2)$$

Therefore, every term factors! We have

$$\begin{aligned}
10^4+324 &= (10^2+2(10)(3)+2(3^2))(10^2-2(10)(3)+2(3^2)) = (178)(58)\\
22^4+324 &= (22^2+2(22)(3)+2(3^2))(22^2-2(22)(3)+2(3^2)) = (634)(370)\\
34^4+324 &= (34^2+2(34)(3)+2(3^2))(34^2-2(34)(3)+2(3^2)) = (1378)(970)\\
46^4+324 &= (46^2+2(46)(3)+2(3^2))(46^2-2(46)(3)+2(3^2)) = (2410)(1858)\\
58^4+324 &= (58^2+2(58)(3)+2(3^2))(58^2-2(58)(3)+2(3^2)) = (3730)(3034)\\
4^4+324 &= (4^2+2(4)(3)+2(3^2))(4^2-2(4)(3)+2(3^2)) = (58)(10)\\
16^4+324 &= (16^2+2(16)(3)+2(3^2))(16^2-2(16)(3)+2(3^2)) = (370)(178)\\
28^4+324 &= (28^2+2(28)(3)+2(3^2))(28^2-2(28)(3)+2(3^2)) = (970)(634)\\
40^4+324 &= (40^2+2(40)(3)+2(3^2))(40^2-2(40)(3)+2(3^2)) = (1858)(1378)\\
52^4+324 &= (52^2+2(52)(3)+2(3^2))(52^2-2(52)(3)+2(3^2)) = (3034)(2410)
\end{aligned}$$

So the fraction becomes

$$\frac{(178)(58)(634)(370)(1378)(970)(2410)(1858)(3730)(3034)}{(58)(10)(370)(178)(970)(634)(1858)(1378)(3034)(2410)} = \frac{3730}{10} = 373$$

21. Note that $4k^4+1$ factors into $(2k^2-2k+1)(2k^2+2k+1)$. Then, by partial fractions, we can see that the expression becomes

$$\sum_{k=1}^{100}\frac{4k}{4k^4+1} = \sum_{k=1}^{100}\left(\frac{1}{2k^2-2k+1}-\frac{1}{2k^2+2k+1}\right)$$

which is *telescoping*. Below is the sum.

$$1-\frac{1}{5}+\frac{1}{5}-\frac{1}{13}+\frac{1}{13}-\frac{1}{25}+\cdots-\frac{1}{19801}+\frac{1}{19801}-\frac{1}{20201} = 1-\frac{1}{20201} = \frac{20200}{20201}$$

22. Let $x=y=z=1$. Then the sum of the coefficients is $3^8=6561$.

23. Letting the expression equal x and squaring results in $x^2=30+x$. Solving, we get $x=6$.

24. We need to factor the sum of three cubes. Let r, s, and t be the three roots. The factoring we use is
$$r^3 + s^3 + t^3 = 3rst + (r + s + t)(r^2 + s^2 + t^2 - (rs + rt + st)).$$
Thus, the quantities on the right side are known from the coefficients! Hence,
$$rst = 210, \quad rs + rt + st = 107, \quad r + s + t = 18, \quad r^2 + s^2 + t^2 = 110.$$
Therefore, $r^3 + s^3 + t^3 = 3(210) + (18)(110 - 107) = 684$.

25. See the *Secrets of Georgia ARML* packet *Making Algebra More Powerful!*

3 The Answers

1. -1
2. 9
3. 223
4. 863
5. $1 - \frac{1}{100!}$
6. 0
7. $n = -6, \ m = -7$
8. 7
9. $\frac{3n^2+5n}{2n^2+6n+4}$
10. 6
11. -2
12. 12
13. $\frac{6}{19}$
14. -50
15. 11
16. $\frac{3}{5}$
17. 11
18. $-1, 0, 1$
19. $21! - 1$
20. 373
21. $\frac{20200}{20201}$
22. 6561
23. 6
24. 684
25. $\frac{7}{8}$