History of Mathematics

A Course for High Schools

Chuck Garner, Ph.D.

Department of Mathematics
Rockdale Magnet School for Science and Technology

NCSSSMST Professional Conference, Nashville, 2010

Outline

SPECIFICS

LENGTH One-semester

PREREQUISITE Calculus (or taken concurrently)

TEXTS Suggested:

- Katz, *A History of Mathematics: an Introduction*, 2nd edition, Addison-Wesley
- Fauvel and Gray, *The History of Mathematics: a Reader*, Open University Press
- Eves, *An Introduction to the History of Mathematics*, 6th edition, Brooks/Cole
- Berlinghoff and Gouvêa, *Math Through the Ages*, 2nd edtition, MAA

Assessment

WEEKENDERS	16 problem sets
READINGS	Classroom discussions
TESTS	Mid-term and final, open book, open notes
PAPER	Biographical research
PROJECT	Textbook

The Ancients

Two weeks

- Origins
- Babylon, Egypt, China
- Computations
- Linear algebraic equations
- Inductive Geometry

Weekender 2, Number 4

The Seqt of a Pyramid The Egyptians measured the steepness of a face of a pyramid by the ratio of the "run" to the "rise"—that is, by giving the reciprocal of what we consider the slope of the face of the pyramid. The vertical unit (the "run") was the *cubit* and the horizontal unit (the "rise") was the *hand*; there were 7 hands in a cubit. With these units, the measure of steepness was called the *seqt* of a pyramid.

Weekender 2, Number 4

(a) Solve Problem 56 of the *Rhind Papyrus* which asks: What is the seqt of a pyramid 250 cubits high and with a square base 360 cubits on a side?
(b) The great pyramid of Cheops has a square base 440 cubits on a side and a height of 280 cubits. What is the seqt of this pyramid?

Weekender 2, Number 5

Sagittas and Chords

(a) Interpret the following, found on a Babylonian tablet dating from 2600 BC:

> *"60 is the circumference, 2 is the sagitta, find the chord."*
> *"Thou, double 2 and get 4, dost thou not see? Take 4 from 20, thou gettest 16. Square 20, thou gettest 400. Square 16, thou gettest 256. Take 256 from 400, thou gettest 144. Find the square root of 144. 12, the square root, is the chord. Such is the procedure."*

WEEKENDER 2, NUMBER 5

SAGITTAS AND CHORDS

(b) Derive a correct formula for the area of a circular segment in terms of the chord c and sagitta s of the segment.

THE GREEKS

Four weeks

- Thales, Pythagoras
- Discovery of irrationals
- Origin of Axiomatics
- Euclid's *Elements*
- Archimedes, Ptolemy, Diophantus

Handout: Ptolemy's Theorem

In a cyclic quadrilateral the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.

Handout: Ptolemy's Theorem

Corollary 1 If a and b are the chords of two arcs of a circle of unit radius, then

$$s = \frac{a}{2}\sqrt{4-b^2} + \frac{b}{2}\sqrt{4-a^2}$$

is the chord of the sum of the two arcs.

Proof.

Let $ABCD$ be a cyclic quadrilateral where AC is a diameter and $s = DB$, $a = BC$, and $b = CD$. Now apply Ptolemy's Theorem. $\square$

Weekender 4, Number 1

DEFINITIONS The definition of a technical term (beyond primitive ones) of a logical discourse serves merely as an abbreviation for a complex combination of terms already present. Thus a new term introduced by definition is really arbitrary, though convenient, and may be entirely dispensed with; but then the discourse would immediately increase in complexity.

Weekender 4, Number 1

(a) Without using the words *diagonals*, *parallels*, or *parallelogram*, restate the proposition: “The diagonals of a parallelogram bisect each other.”

(b) By means of appropriate definitions, reduce the the following sentence to one containing no more than five words: “The movable seats with four legs were restored to a sound state by the person who takes care of the building.”

Weekender 6, Number 4

Problems from Diophantus A common technique employed by Diophantus to solve a system of equations is the following. Consider the system

$$\begin{cases} x + y = a \\ x^2 + y^2 = b. \end{cases}$$

Diophantus sets $x = \frac{a}{2} + z$ and $y = \frac{a}{2} - z$. (Note that this ensures that $x + y = a$.) He then substitutes these expressions in the second equation; this results in a single quadratic in the new variable z, whose solution is straightforward. Having found z, it is then easy to produce the values of x and y.

The Middle Ages

Two weeks

- Evolution of algebra
- Evolution of numerals
- Islam, China, India
- Fibonacci

HANDOUT: PROBLEMS FROM THE *Greek Anthology*

A DISTRIBUTION PROBLEM How many apples are needed if four persons of six receive one-third, one-eighth, one-fourth, and one-fifth, respectively, of the total number, while the fifth receives ten apples, and one apple remains left for the sixth person?

Handout: Problems from the *Greek Anthology*

An age problem Demochares has lived a fourth of his life as a boy, a fifth as a youth, a third as a man, and has spent 13 years in his dotage. How old is he?

HANDOUT: PROBLEMS FROM THE *Greek Anthology*

A WORK PROBLEM Brickmaker, I am in a hurry to erect this house. Today is cloudless, and I do not require many more bricks, for I have all I want but three hundred. Thou alone couldst make as many, but thy son left off working when he had finished two hundred, and thy son-in-law when he had made two hundred and fifty. Working all together, in how many days can you make these?

Weekender 8, Number 2

THE MATHEMATICAL POPE Gerbert (950-1003) was born in France and from an early age, he showed amazing abilities. He was one of the first Christians to study in the Moslem schools in Spain, and may have brought back the Hindu-Arabic numerals to Christian Europe. He is said to have constructed abaci, terrestrial and celestial globes, a clock, and reportedly an organ. He was elected Pope as Sylvester II in 999. The following problem is from Gerbert's *Geometry*, and was considered quite difficult at the time. Determine the legs of a right triangle whose hypotenuse is 5 and area is $3\frac{9}{25}$.

Weekender 8, Number 4

THE BROKEN BAMBOO As a small example of how similar problems occur in different cultures, solve the following two "broken bamboo" problems.

(a) A bamboo 18 cubits high was broken by the wind. Its top touched the ground 6 cubits from the root. Tell the lengths of the segments of bamboo. *(Brahmagupta, 628)*

(b) There is a bamboo 10 *ch'ih* high, the upper end of which being broken reaches the ground 3 *ch'ih* from the stem. Find the height of the break. *(Nine Sections, 1261)*

Weekender 8, Number 6

Ten Into Two Parts As a small example of how similar problems occur in different cultures, solve the following two "broken bamboo" problems.

(b) "I have divided 10 into two parts; I multiply the one by 10 and the other by itself, and the products were the same. Find the two parts." *(al-Khwarizmî, c.825)*

(d) "One says that 10 is divided into two parts, each of which is divided by the other, and when each of the quotients is multiplied by itself and the smaller is subtracted from the larger, then there remains 2. Find the two parts." *(Abū Kamīl, c.900)*

The Renaissance

Two weeks

- Solution of cubics
- Logarithms
- Astronomy
- Analytic geometry
- Probability

Handout: Solutions to Cubics

Cardano solves $x^3 + mx = n$ by considering the identity

$$(a-b)^3 + 3ab(a-b) = a^3 - b^3.$$

If we choose a and b so that $3ab = m$ and $a^3 - b^3 = n$, then x is given by $a - b$. But, solving the last two equations for a and b, one has

$$a = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}, \quad b = \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

so that

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}$$

Weekender 9, Number 3

CUBIC TRANSFORMATION The transformation

$$x = z - \frac{b}{3a}$$

converts the general cubic

$$ax^3 + bx^2 + cx + d = 0$$

into one of the form

$$z^3 + 3Hz + G = 0.$$

(a) Find H and G in terms of a, b, c, and d.
(b) Transform $x^3 + 21x = 9x^2 + 5$; use Cardano's Formula to solve the reduced equation; solve the given equation.

WEEKENDER 9, NUMBER 7

STRANGE REMARK Explain the remark in Galileo's *The Two New Sciences* that

> *"neither is the number of squares less than the totality of all numbers, nor the latter greater than the former."*

Analysis

Two weeks

- Calculus
- Euler
- Lagrange, Laplace, Legendre, Fourier
- Gauss

Weekender 11, Number 10

Euler's Derangements Euler was posed the following problem: "Given any series of n letters a, b, c, d, etc., to find how many ways they can be arranged so that none returns to the position it initially occupied." Euler called this a *derangement*. If $\Pi(n)$ is the number of derangements of n objects, then

$$\Pi(1) = 0, \quad \Pi(2) = 1, \quad \Pi(3) = 2, \quad \Pi(4) = 9, \quad \Pi(5) = 44.$$

Euler discovered a general formula:

$$\Pi(n) = n!\left(\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right)$$

(a) Find $\Pi(6)$ and $\Pi(7)$.
(b) Given any series of 7 letters, find the probability that none returns to its original position.

Handout: Gauss' Law of Quadratic Reciprocity

First, some notation: the *Legendre symbol* $(\frac{p}{q})$ is defined to be 1 if p is a quadratic residue mod q, and -1 if not. Legendre proved the straight-forward product rule for the Legendre symbol: $(\frac{ab}{q}) = (\frac{a}{q})(\frac{b}{q})$.

Theorem If p and q are distinct odd primes, then $(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}$.

Handout: Gauss' Law of Quadratic Reciprocity

The more useful form of this theorem is $\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right)(-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}$.
To determine whether 19 is or is not a quadratic residue mod 61, we compute the Legendre symbols, reducing by mods each time:

$$\left(\frac{19}{61}\right) = \left(\frac{61}{19}\right)(-1)^{30\cdot 9} = \left(\frac{61}{19}\right) = \left(\frac{4}{19}\right) = \left(\frac{2}{19}\right)\left(\frac{2}{19}\right) = 1$$

WEEKENDER 12, NUMBER 3

GAUSSIAN INTEGERS Which of the following real primes are primes as Gaussian integers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37? Demonstrate a factorization of those that are composite Gaussian integers.

Abstraction

Two weeks

- Non-Euclidean geometry
- Non-commutative algebra
- Groups
- The *Erlanger Programme*

Weekender 13, Number 6

Snarks and Trones Consider the following set of postulates about certain objects called *trones*, and certain collections of trones called *snarks*.

1. Every snark is a collection of trones.
2. There exist at least two trones.
3. If p and q are two trones, then there exists exactly one snark containing both p and q.
4. If M is a snark, then there exists a trone not in M.
5. If M is a snark, and p is a trone not in M, then there exists exactly one snark containing p and not containing any trone that is in M.

Weekender 13, Number 6

(a) Interpret snark as "committee" and trone as "committee member." Are the postulates consistent?
(b) Interpret trone as any one of the three letters a, b, c, and snark as any one of the three pairs ab, ac, bc. Are the postulates consistent?
(c) Consider the postulates under the following interpretation: a trone is a point and a snark is a line.

Handout: Examples of New Sums and Products

4. Let S be the set of all real numbers of the form $m+n\sqrt{2}$, where m and n are integers, and let $+$ and $\times$ denote the usual addition and multiplication of real numbers.
5. Let S be the set of Gaussian integers, and let $+$ and $\times$ denote the usual addition and multiplication of complex numbers.
8. Let S be the set of all ordered pairs (m,n) of integers, and let $(a,b)+(c,d)=(a+c,b+d)$ and $(a,b)\times(c,d)=(ac-bd,ad+bc)$.
12. Let S be the set of all point sets of the plane, and let $a+b$ denote the union of sets a and b, and $a\times b$ the intersection of sets a and b. As a special point set of the plane we introduce an ideal set, the *null set*, which has no points in it.

Weekender 13, Number 7

BINARY OPERATIONS Determine whether the following binary operations $*$ and $\diamond$, defined for the positive integers, obey the commutative and associative laws, and whether the operation $\diamond$ is distributive over the operation $*$.

1. $a * b = a + 2b, \quad a \diamond b = 2ab.$
2. $a * b = a + b^2, \quad a \diamond b = ab^2.$
3. $a * b = a^b, \quad a \diamond b = b.$
4. $a * b = a^2 + b^2, \quad a \diamond b = a^2 b^2.$

FOUNDATIONS

Two weeks

- Formal axiomatics
- Cantor’s set theory
- Metamathematics
- Gödel’s incompleteness theorem

Weekender 14, Number 7

Denumerable Sets Prove the following.

(a) The union of a finite number of denumerable sets is a denumerable set.

(b) The union of a denumerable number of denumerable sets is a denumerable set.

WEEKENDER 15, NUMBER 1

INTUITION VS. ABSTRACTION One purpose of formal axiomatics, with its abstraction and symbolism, is to furnish a barrier against the use of intuition. Answer the following questions intuitively, and then check your answers by calculation.

(b) A man sells half his apples at 3 for 17 cents and then sells the other half at 5 for 17 cents. At what rate should he sell all his apples in order to make the same income?

(d) A clock strikes six in 5 seconds. How long will it take to strike twelve?

Weekender 15, Number 1

(e) A bottle and a cork together cost \$1.10. If the bottle costs a dollar more than the cork, how much does the cork cost?
(f) Two jobs have a starting salary of \$60,000 per year with salaries paid every six months. One job offers an annual raise of \$8000 and the other job offers a semiannual raise of \$2000. Which is the better paying job?
(h) Four-fourths exceeds three-fourths by what fractional part?

Weekender 15, Number 2

Valid Conclusions In each of the following, is the given conclusion a valid deduction from the given premises?

(a) If today is Saturday, then tomorrow will be Sunday.
But tomorrow will be Sunday.
Therefore, today is Saturday.

(c) If a is b, then c is d.
But c is d.
Therefore, a is b.

Handout: Abstract Spaces

In 1906, Maurice Fréchet introduced the concept of a *metric space.*

Definition. A *metric space* is a set M of elements called *points*, together with a real number $d(x,y)$ called the *distance function*, or *metric*, of the space, associated with each ordered pair of points x and y in M, and that satisfies the following four postulates:

M1. $d(x,y) \geq 0$

M2. $d(x,y) = 0$ if and only if $x = y$

M3. $d(x,y) = d(y,x)$

M4. $d(x,z) \leq d(x,y) + d(y,z)$, where x, y, and z are any three points in M. (This is called the *triangle inequality*.)