Problem Solving Strategies for Math Teams
(Or, How to Count)

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"Calculus is easy. Counting is hard."

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More Than Meets The Eye

Outline

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Extra-Curricular Math

- Number Theory
- Graph Theory
- Logic and Proofs
- Advanced Algebra
- Advanced Euclidean Geometry
- Geometry of the Complex Plane

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- Graph Theory
- Logic and Proofs
- Advanced Algebra
- Advanced Euclidean Geometry
- Geometry of the Complex Plane
- Combinatorics

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1. Introduction
2. **Basic Counting Strategies**
3. Problems
4. Further Reading

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Addition and Multiplication

Addition There are m varieties of soup and n varieties of salad. So there are $m + n$ ways to eat soup *or* salad (but not both soup and salad).

Multiplication The number of ways to eat both soup *and* salad is mn.

Permutations

Permutation There are n different (distinct) objects. The number of ways to reorder them is $n!$.

Permutations of subsets There are n distinct objects. The number of ways to order r of them (where $r \leq n$) is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

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Combinations

The Mississippi Formula There are 11 letters in the word *MISSISSIPPI*. The number of distinct orderings of those letters is

$$\frac{11!}{4!4!2!}.$$

Suppose we have a collection of n marbles that are indistinguishable except for color. The numbers of colors are $c_1, c_2, \ldots, c_k$. Then the number of distinct orderings of the marbles is

$$\frac{n!}{c_1!c_2!\cdots c_k!}.$$

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Combinations

The Pizza Formula There are eight pizza toppings available and we get to pick three of them to put on a pizza. The number of pizzas we can make is

$$\frac{8!}{3!5!}.$$

There are n distinct objects, and we wish to pick r of them (where $r \leq n$) where in which we pick the r objects does not matter. The number of ways to do this is

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Summary

The number of ways to choose pasta and sauce from 7 pastas and 6 sauces	7×6
The number of ways to choose a meal from 12 beef dinners, 6 chicken dinners or 5 vegetarian dinners	$12 + 6 + 5$
The number of ways you can answer seven multiple-choice questions with five choices per question	5^7
The number of ways of choosing a team of three people from a group of 12	$\binom{12}{3}$

Summary

The number of ways of choosing a club president, vice-president, and secretary, from the 12 members	$P(12,3)$
The number of ways of choosing a team of 3 people from a group of 12 and picking a leader of the team	$3\binom{12}{3}$
The number of ways of choosing 3 boys and 8 girls from a group of 12 boys and 10 girls	$\binom{12}{3}\binom{10}{8}$
The number of ways of picking a team of either 3 or 4 people from a group of 12	$\binom{12}{3}+\binom{12}{4}$

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Mississippi Mathematics

A typesetters apprentice, who is carrying a tray of letters forming the word *mathematics*, trips and spills the letters on the floor. If the apprentice randomly rearranges the letters into an eleven-letter word, what is the probability that the result will again be *mathematics*?

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The total number of distinct rearrangements is

$$\frac{11!}{2!2!2!} = \frac{11!}{8} = 11 \cdot 10 \cdot 9 \cdot 7! = 990 \cdot 7!$$

There are 8 ways to obtain *mathematics*. Probability is then

$$\frac{8}{990 \cdot 7!} = \frac{1}{990 \cdot 7 \cdot 6 \cdot 5 \cdot 3} = \frac{1}{990 \cdot 630} = \frac{1}{623700}$$

Seating Arrangements I

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$17!$

Seating Arrangements II

Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if all the boys sit next to each other and all the girls sit next to each other?

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Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if all the boys sit next to each other and all the girls sit next to each other?
There are 8! ways for the boys and 9! ways for the girls, but the groups could sit boys-girls or girls-boys. So there are $2 \cdot 8! \cdot 9!$ ways.

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Seating Arrangements III

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They must sit girl-boy-girl-boy- etc., so there are $9! \cdot 8!$ ways.

Seating Arrangements IV

Ann, Bill, and Carol sit on a row of 6 seats. If no two of them sit next to each other, in how many different ways can they be seated?

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The three can sit in 3! ways, but the empty seats can be picked in two ways:

E O E O E O or O E O E O E

for a total of $2 \cdot 3! = 12$ ways.

Seating Arrangements V

Adam, Ben, Charlie, Chuck, Debbie, Don, Steve, and Tom are to sit around a circular table. In how many distinct arrangements can they sit around the table if Don and Debbie must sit next to each other?

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First, treat Don and Debbie as one person ("Dondebbie"). Next, sit one person, say Adam, down and don't move him (this prevents overcounting due to rotational symmetry). Then fill in the others around him. There are six people left, so there are $6! = 720$ ways...

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Gift Giving I

We have three different toys and we want to give them away to two girls and one boy (one toy per child). The children will be selected from four boys and six girls. In how many ways can this be done?

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First Solution:

There are $\binom{4}{1} = 4$ ways to pick a boy and $\binom{6}{2} = 15$ ways to pick 2 girls. Since toys are different, they can be given in $3! = 6$ ways. Total: $4 \cdot 15 \cdot 6 = 360$.

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Gift Giving I

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Second Solution:

There are $\binom{4}{1} = 4$ ways to pick a boy and $\binom{3}{1} = 3$ ways to pick a toy for him. Then pick two girls in order so that the first girl gets the second toy and the second girl gets the third toy: $P(6, 2) = 30$. Total: $4 \cdot 3 \cdot 30 = 360$.

Gift Giving II

We have three different toys and we want to give them away to three kids (one toy per child). The children will be selected from four boys and six girls. In how many ways can this be done if at least two boys get a toy?

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Required: Count ways for two boys; count ways for three boys; add them.

Gift Giving II

We have three different toys and we want to give them away to three kids (one toy per child). The children will be selected from four boys and six girls. In how many ways can this be done if at least two boys get a toy?

Required: Count ways for two boys; count ways for three boys; add them.

Two boys: There are $\binom{4}{2} = 6$ ways for boys, $\binom{6}{1} = 6$ for a girl, and we give out toys in $3! = 6$ ways; this total is 216.

Three boys: There are $P(4, 3) = 24$ ways since order matters.

Overall total: $216 + 24 = 240$.

Path Problem I

Peter Rabbit starts hopping at the point $(0, 0)$ and finishes at $(6, 0)$. Each hop is from a point of the form (x, y) to one of the points $(x + 1, y \pm 1)$. However, he is not allowed to jump on a point (x, y) with $y < 0$ (because there is poison ivy in the fourth quadrant). How many paths are there from $(0, 0)$ to $(6, 0)$?

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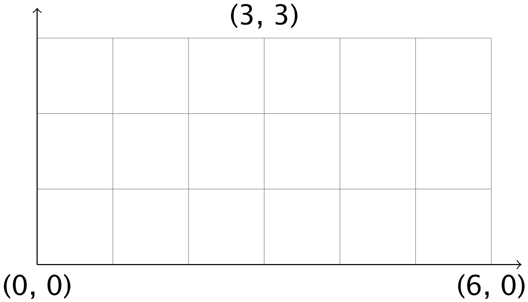
Path Problem I



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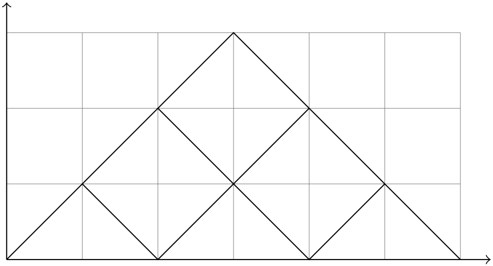

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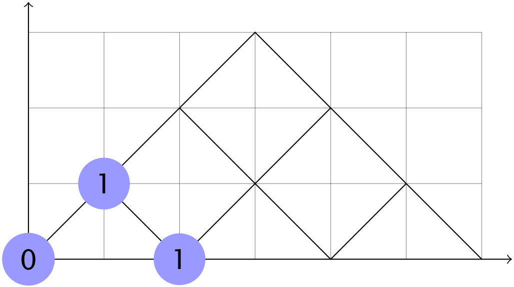

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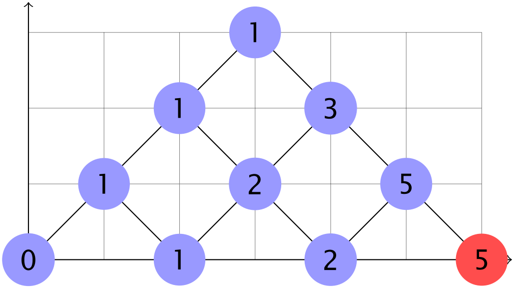
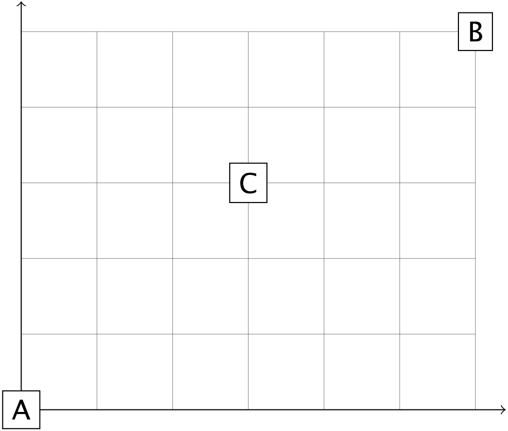


Path Problem II

Starting at the point $A(0, 0)$, we are required to move only up or to the right to reach the point $B(6, 5)$. However, we must avoid the point $C(3, 3)$. How many ways are there for us to make this journey?

First solution. . .

Path Problem II



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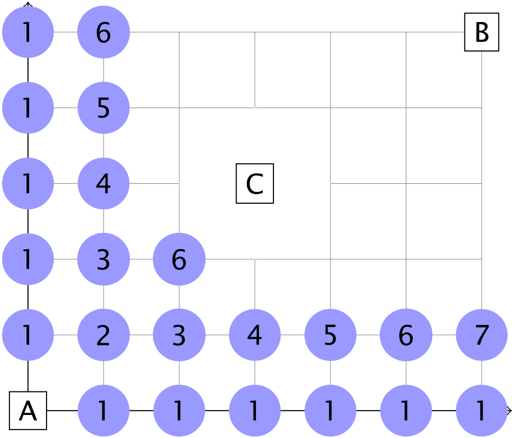

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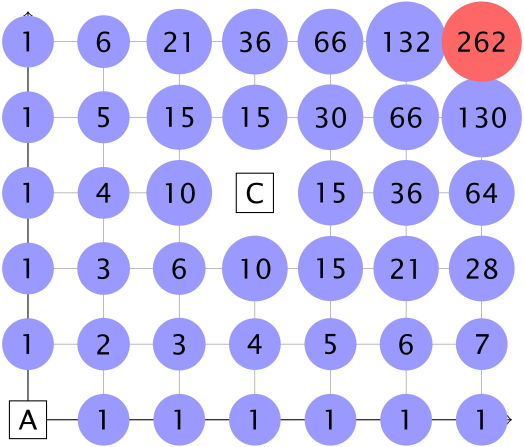

Path Problem III

Starting at the origin, draw a path to the point $(10, 10)$ that stays on the unit gridlines and has a total length of 20. One possible path is

$$(0, 0) \to (0, 7) \to (4, 7) \to (4, 10) \to (10, 10).$$

How many different paths are there?

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Using R for right and U for up, the suggested path can be encoded by

$$UUUUUUURRRRUUURRRRRR.$$

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How many different paths are there?
Using R for right and U for up, the suggested path can be encoded by

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Any path will include exactly 10 Rs and 10 Us. So the total must be

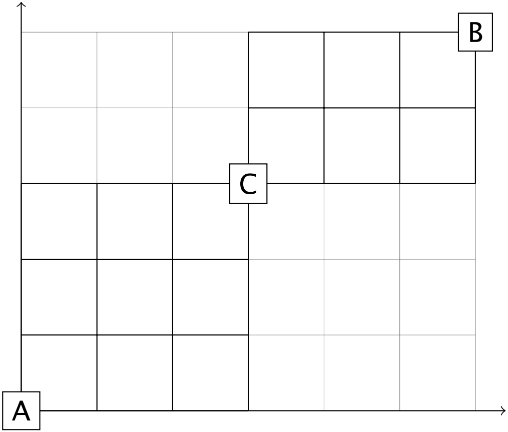
$$\binom{20}{10} = 184,756.$$

Path Problem II Revisited

Starting at the point $A(0,0)$, we are required to move only up or to the right to reach the point $B(6,5)$. However, we must avoid the point $C(3,3)$. How many ways are there for us to make this journey?

Second solution...

Path Problem II Revisited



Path Problem II Revisited

Starting at the point $A(0, 0)$, we are required to move only up or to the right to reach the point $B(6, 5)$. However, we must avoid the point $C(3, 3)$. How many ways are there for us to make this journey?

Second solution...

Paths from A to C: $\binom{6}{3} = 20$; paths from C to B: $\binom{5}{2} = 10$.

Paths from A to B that go through C: $20 \times 10 = 200$

Paths from A to B: $\binom{11}{5} = 462$.

Thus, the number of paths that avoid C is $462 - 200 = 262$.

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Binomial Counting I

Find the coefficient of x^4y in the expansion of $(2x - 3y)^5$.

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Binomial Counting I

Find the coefficient of x^4y in the expansion of $(2x - 3y)^5$.
Picking four xs and one y from five things can be done in $\binom{5}{4} = \binom{5}{1} = 5$ ways. But in picking four xs and one y, we also pick four 2s and one -3. Hence, the coeffecient is $5 \cdot 2^4 \cdot -3 = -240$.

Binomial Counting II

Find the coefficient of the $a^6 b^6 c^2$ term in the expansion of $\left[a + (b + c)^4\right]^8$.

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In order to get a^6, we must pick two $(b + c)^4$ from eight things; this is done in $\binom{8}{2} = 28$ ways.

Binomial Counting II

Find the coefficient of the $a^6 b^6 c^2$ term in the expansion of $[a + (b + c)^4]^8$.

In order to get a^6, we must pick two $(b + c)^4$ from eight things; this is done in $\binom{8}{2} = 28$ ways. The two we pick are of the form $[(b + c)^4]^2 = (b + c)^8$. Here, we need six bs and two cs from eight things, which can be picked in $\binom{8}{2} = 28$ ways. Hence the coefficient is $28^2 = 784$.

Stars and Bars I

I have seven indistinguishable ping pong balls that are to be placed in 3 different urns. In how many different ways may I fill the urns? (Some urns may remain empty.)

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Encode a possible placement of the balls in the urns using stars and bars. The stars represent the balls and we use separators for the urns; i.e., the placement of 2 balls in urn 1, 5 in urn 2, and none in urn 3 is encoded by

$$\star\ \star\ \mid\ \star\ \star\ \star\ \star\ \star\ \mid\mid$$

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$$\star\ \star\ |\ \star\ \star\ \star\ \star\ \star\ |\ |$$

Moving the bar separators around gives a different placement of balls in the urns. Hence, we pick the 2 separators from among 7 stars and 2 bars ($7 + 2 = 9$) to get the number of ways as $\binom{9}{2} = 36$.

Stars and Bars II

How many positive integer solutions are there for the equation $x + y + z = 50$?

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How many positive integer solutions are there for the equation $x + y + z = 50$?

The 50 positive integers are the stars and we separate them into three "urns" x, y, and z. But we require positive integer solutions, so none of the "urns" may remain empty.

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Stars and Bars II

How many positive integer solutions are there for the equation $x + y + z = 50$?

The 50 positive integers are the stars and we separate them into three “urns” x, y, and z. But we require positive integer solutions, so none of the “urns” may remain empty. Hence, we drop one star into each “urn” leaving 47 stars. The number solutions is

$$\binom{47+2}{2} = \binom{49}{2} = 1176.$$

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Stars and Bars III

How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$?

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How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$?

Note x must be odd; we write $x = 2k + 1$ and the equation becomes

$$2k + 1 + 2y + 2z = 99 \qquad \text{or} \qquad k + y + z = 49.$$

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Stars and Bars III

How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$?

Note x must be odd; we write $x = 2k + 1$ and the equation becomes

$$2k + 1 + 2y + 2z = 99 \qquad \text{or} \qquad k + y + z = 49.$$

The solutions are nonnegative so some "urns" may remain empty. Thus,

$$\binom{49+2}{2} = \binom{51}{2} = 1275.$$

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Further Reading

- Paul Zeitz, *The Art and Craft of Problem Solving*
- David Patrick, *Introduction to Counting and Probability* and *Intermediate Counting and Probability*
- Titu Andreescu and Zuming Feng, *A Path to Combinatorics for Undergraduates* and *102 Combinatorial Problems*
- Chen Chuan-Chong and Koh Khee-Meng, *Principles and Techniques in Combinatorics*