Mathematical Problem Solving with Math Teams

(Session #171)

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Outline

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Math Team Math

Four major categories of Math Team math

- Number Theory
- Algebra
- Geometry
- Combinatorics

Solve these problems.

1. Find the coefficient of the x^5y^3 term when $(x+y)^8$ is expanded.
2. How many ways can one select a committee of 3 people from a group of 8 people?
3. How many paths are there from the origin to the point $(3,5)$ moving only moving north and east along gridlines?
4. How many nonnegative integer solutions are there to the equation $w+x+y+z=5$?

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What do these problems have in common?

Outline

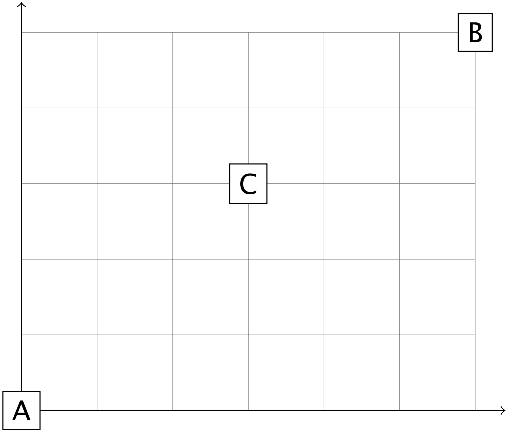
Seating Arrangements

1. Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there?
2. Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if all the boys sit next to each other and all the girls sit next to each other?
3. Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if no child sits next to a child of the same gender?

Path Problem I

Starting at the point $A(0, 0)$, we are required to move only up or to the right to reach the point $B(6, 5)$. However, we must avoid the point $C(3, 3)$. How many ways are there for us to make this journey?

Path Problem I



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First solution. . .

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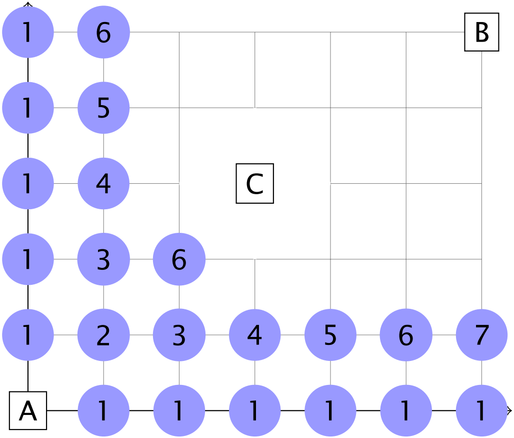

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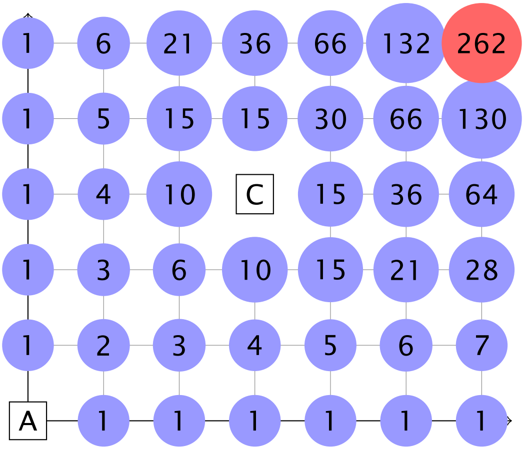

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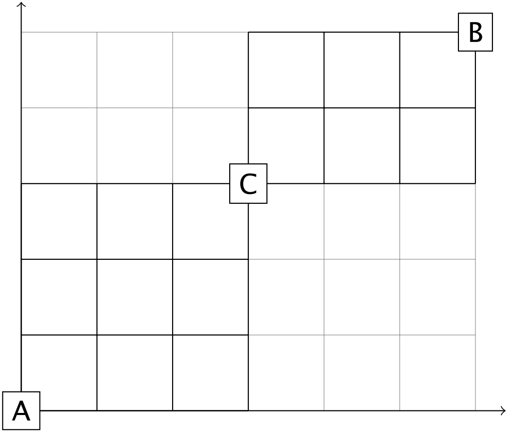

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Second solution. . .

Paths from A to C: $\binom{6}{3} = 20$; paths from C to B: $\binom{5}{2} = 10$.

Paths from A to B that go through C: $20 \times 10 = 200$

Paths from A to B: $\binom{11}{5} = 462$.

Thus, the number of paths that avoid C is $462 - 200 = 262$

Path Problem II

Starting at the origin, draw a path to the point $(6,6)$ that stays on the unit gridlines and has a total length of 20. One possible path is

$$(0,0) \to (0,2) \to (5,2) \to (5,6) \to (6,6).$$

How many different paths are there?

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Any path will include exactly 6 Rs and 6 Us. So the total must be

$$\binom{12}{6} = 924.$$

Binomial Counting I

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Notice that

$$(2x-3y)^5 = (2x-3y)(2x-3y)(2x-3y)(2x-3y)(2x-3y).$$

Picking four $2x$s and one $-3y$ from five things can be done in $\binom{5}{4} = \binom{5}{1} = 5$ ways. But in picking four $2x$s and one $-3y$, we also have the product of four 2s and one -3. Hence, the coeffecient is $5 \cdot 2^4 \cdot -3 = -240$.

Stars and Bars I

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Encode a possible placement of the balls in the urns using stars and bars. The stars represent the balls and we use separators for the urns; i.e., the placement of 2 balls in urn 1, 5 in urn 2, and none in urn 3 is encoded by

$$\star\ \star\ |\ \star\ \star\ \star\ \star\ \star\ |$$

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Moving the bar separators around gives a different placement of balls in the urns. Hence, we pick the 2 separators from among 7 stars and 2 bars $(7 + 2 = 9)$ to get the number of ways as $\binom{9}{2} = 36$.

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The 50 positive integers are the stars and we separate them into three “urns” x, y, and z. But we require positive integer solutions, so none of the “urns” may remain empty. Hence, we drop one star into each “urn” leaving 47 stars. The number solutions is

$$\binom{47+2}{2} = \binom{49}{2} = 1176.$$

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Note x must be odd; we write $x = 2k + 1$ and the equation becomes

$$2k + 1 + 2y + 2z = 99 \qquad \text{or} \qquad k + y + z = 49.$$

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The solutions are nonnegative so some “urns” may remain empty. Thus,

$$\binom{49+2}{2} = \binom{51}{2} = 1275.$$

Outline

Solve these problems.

1. Find the positive solution to the equation $2x^4 + x^3 + x + 2 = 6$.
2. When $(29x^2 - 28y^2)^{10}$ is expanded, what is the sum of the coefficients?
3. Compute $\sum_{k=0}^{10} \binom{10}{k}$.

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3. Compute $\sum_{k=0}^{10} \binom{10}{k}$.
4. Convert the base-3 number 21012_3 into base 10.
5. Convert the base-3 number 21012_3 into base 9.
6. Subtract: $21012_4 - 21012_3$.

Outline

Solve these problems.

1. Find the remainder when 89 is divided by 17.
2. Find the remainder when 89^{16} is divided by 17.
3. Find the units digit of 2017^{1000}.
4. Find the last two digits of 2017^{1000}.
5. A *primitive Pythagorean triple* is a Pythagorean triple (a, b, c) where a, b and c do not share a common factor other than 1. Prove that the hypotenuse of a primitive Pythagorean triple is never divisible by 3.

Some Number Theory Facts

- Definition of mod $a \equiv b \bmod n$ if and only if $n \mid (b - a)$.
- Fermat's Little Theorem $a^{p-1} \equiv 1 \bmod p$, where p is prime and doesn't divide a. (The remainder when a^{p-1} is divided by p is 1.)
- Euler's totient function $\phi(n)$, is the number if positive integers less than and relatively prime to n.
- Euler's Theorem $a^{\phi(n)} \equiv 1 \bmod n$, where a and n are relatively prime. (The remainder when $a^{\phi(n)}$ is divided by n is 1.)

Modular Arithmetic

Compute the following.

1. $89 \bmod 7$
2. $711 \bmod 10$
3. $1000 \bmod 11$
4. $\phi(23)$, $\phi(10)$, $\phi(100)$
5. $16^{16} \bmod 17$
6. $19^{16} \bmod 17$
7. $19^{17} \bmod 17$
8. $67^{83} \bmod 17$
9. $303^{2016} \bmod 100$
10. $13^{82} \bmod 100$

Modular Problem

Compute $898^{2016} \bmod 100$.

Modular Problem

Compute 898^{2016} mod 100.

Since $\phi(100) = 40$, we can replace the exponent of 2016 with its remainder with 2016 is divided by 40; this is 16. Hence, we compute 898^{16} mod 100. Since $100 \mid (898 - (-2))$ we have that $898 \equiv -2 \bmod 100$. Hence, we can replace 898 with -2, and compute $(-2)^{16}$ mod 100. Finally,

$$(-2)^{16} = 2^{16} = 2^{10}2^{6}$$

so

$$\begin{aligned} 898^{2016} \bmod 100 &\equiv 1024 \cdot 64 \bmod 100 \\ &\equiv 24 \cdot 64 \bmod 100 \\ &\equiv 36 \bmod 100. \end{aligned}$$

Challenges!

1. Find the last two digits of 2016^{2016}.
2. Find the remainder when 2016^{2017} is divided by 41.

Challenges!

1. Prove that the square of an integer gives a remainder of 0 or 1 when divided by 4.
2. Prove that the area of a right triangle with integer sides is even.
3. Prove that the cube of an even integer gives a remainder of 0 when divided by 8.
4. Suppose p is a prime larger than 3. Prove that $p^2 - 1$ is divisible by 12.
5. Find all solutions, in integers, to the equation $x^2 + y^2 = 2015$.

Questions?

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This was session #171