# Counting Strategies for Math Teams

Chuck Garner, PhD
Rockdale Magnet School for Science and Technology
Conyers, Georgia
cgarner@rockdale.k12.ga.us

October 20, 2011

Counting can be very difficult. But a few good methods can go a long way to solving counting problems.

**Addition** There are $m$ varieties of soup and $n$ varieties of salad. So there are $m + n$ ways to eat soup *or* salad (but not both soup and salad).

**Multiplication** The number of ways to eat both soup *and* salad is $mn$.

**Permutation** There are $n$ different (distinct) objects. The number of ways to reorder them is $n!$.

**Permutations of subsets** There are $n$ distinct objects. The number of ways to order $r$ of them (where $r \leq n$) is

$$P(n, r) = \frac{n!}{(n-r)!}.$$

**The Mississippi Formula** There are 11 letters in the word MISSISSIPPI. The number of different (distinct) orderings of those letters is

$$\frac{11!}{4!4!2!}.$$

Suppose we have a collection of $n$ marbles that are indistinguishable except for color. The numbers of colors are $c_1$, $c_2$, $\ldots$, $c_k$. Then the number of distinct orderings of the marbles is

$$\frac{n!}{c_1!c_2!\cdots c_k!}.$$

**The Pizza Formula** There are eight pizza toppings available and we get to pick three of them to put on a pizza. (To pick olives-mushroom-sausage results in the same pizza as mushroom-sausage-olives.) The number of pizzas we can make is then

$$\frac{8!}{3!5!}.$$

There are $n$ distinct objects, and we wish to pick $r$ of them (where $r \leq n$). The order in which we pick the $r$ objects does not matter. Then the number of ways to do this is

$$C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

**Common English-to-Math Counting**

| | |
|---|---|
| The number of ways to choose pasta and sauce from 7 pastas and 6 sauces | $7 \times 6$ |
| The number of ways to choose a meal from 12 beef dinners, 6 chicken dinners or 5 vegetarian dinners | $12 + 6 + 5$ |
| The number of ways you can answer seven multiple-choice questions with five choices per question | $5^7$ |
| The number of ways of choosing a team of three people from a group of 12 | $\binom{12}{3}$ |
| The number of ways of choosing a club president, vice-president, and secretary, from the 12 members | $P(12,3)$ |
| The number of ways of choosing a team of 3 people from a group of 12 and picking a leader of the team | $3\binom{12}{3}$ |
| The number of ways of choosing 3 boys and 8 girls from a group of 12 boys and 10 girls | $\binom{12}{3}\binom{10}{8}$ |
| The number of ways of picking a team of either 3 or 4 people from a group of 12 | $\binom{12}{3} + \binom{12}{4}$ |

## Problems

1. **Mississippi Mathematics** A typesetters apprentice, who is carrying a tray of letters forming the word MATHEMATICS, trips and spills the letters on the floor. If the apprentice randomly rearranges the letters into an eleven-letter word, what is the probability that the result will again be MATHEMATICS? 2002 Mercer University

2. **Seating Arrangements I** Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there?
Zeitz, *The Art and Craft or Problem Solving*

3. **Seating Arrangements II** Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if all the boys sit next to each other and all the girls sit next to each other? Zeitz, *The Art and Craft or Problem Solving*

4. **Seating Arrangements III** Eight boys and nine girls sit in a row of 17 seats. How many different seating arrangements are there if no child sits next to a child of the same gender? Zeitz, *The Art and Craft or Problem Solving*

5. **Seating Arrangements IV** Ann, Bill, and Carol sit on a row of 6 seats. If no two of them sit next to each other, in how many different ways can they be seated?
2007 Australian Math Contest, Junior Division

6. **Seating Arrangements V** Adam, Ben, Charlie, Chuck, Debbie, Don, Steve, and Tom are to sit around a circular table. In how many distinct arrangements can they sit around the table if Don and Debbie must sit next to each other?
2008 GCTM State Tournament

7. **Gift Giving I** We have three different toys and we want to give them away to two girls and one boy (one toy per child). The children will be selected from four boys and six girls. In how many ways can this be done?
Zeitz, *The Art and Craft or Problem Solving*

8. **Gift Giving II** We have three different toys and we want to give them away to three kids (one toy per child). The children will be selected from four boys and six girls. In how many ways can this be done if at least two boys get a toy?
Zeitz, *The Art and Craft or Problem Solving*

9. **Path Problem I** Peter Rabbit starts hopping at the point $(0,0)$ and finishes at $(6,0)$. Each hop is from a point of the form $(x,y)$ to one of the points $(x+1, y \pm 1)$. However, he is not allowed to jump on a point $(x,y)$ with $y < 0$ (because there is poison ivy in the fourth quadrant). How many paths are there from $(0,0)$ to $(6,0)$? 2008 GCTM State Tournament

10. **Path Problem II** Starting at the point $A(0,0)$, we are required to move only up or to the right to reach the point $B(6,5)$. However, we must avoid the point $C(3,3)$. How many ways are there for us to make this journey?

11. **Path Problem III** Starting at the origin, draw a path to the point $(10, 10)$ that stays on the unit gridlines and has a total length of 20. One possible path is

$$(0, 0) \to (0, 7) \to (4, 7) \to (4, 10) \to (10, 10).$$

How many different paths are there? Zeitz, *The Art and Craft or Problem Solving*

12. **Binomial Counting I** Find the coefficient of $x^4y$ in the expansion of $(2x - 3y)^5$.

13. **Binomial Counting II** Find the coefficient of the $a^6b^6c^2$ term in the expansion of $[a + (b + c)^4]^8$. 2007 GACS Math Tournament, Varsity Division

14. **Stars and Bars I** I have seven indistinguishable ping pong balls that are to be placed in 3 different urns. In how many different ways may I fill the urns? (Some urns may remain empty.)

15. **Stars and Bars II** How many positive integer solutions are there for the equation $x + y + z = 50$?

16. **Stars and Bars III** How many nonnegative solutions are there to the equation $x + 2y + 2z = 99$? 2008 Georgia ARML practice

## Further Reading

- Paul Zeitz, *The Art and Craft of Problem Solving*, second edition, John Wiley & Sons ($48)
- David Patrick, *Introduction to Counting and Probability* and *Intermediate Counting and Probability*, Art of Problem Solving ($38 and $42)
- Titu Andreescu and Zuming Feng, *A Path to Combinatorics for Undergraduates* and *102 Combinatorial Problems*, Birkhäuser ($55 and $50)
- Chen Chuan-Chong and Koh Khee-Meng, *Principles and Techniques in Combinatorics*, World Scientific ($37)