Teaching Integration

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October 9, 2013 / Georgia DOE AP Content Workshop
Bainbridge High School

Outline

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1. **Integration Then and Now**
2. **Problems**
   - Areas
   - Antiderivatives
   - The Fundamental Theorem of Calculus
   - Applications
3. **Logarithms**
4. **Summary**

How I Used To Do Things

- Began with the antiderivative
- Substitution
- “Integral Means Area”
- Riemann sums; Sigma notation
- Fundamental Theorem
- Integrals Involving Logarithms/Exponentials
- Areas
- Trapezoids
- Volume
- Differential Equations

Things I Noticed Using This Approach

- No reason for the integral to exist
- Heavy on symbolic manipulation
- Connection with area from out of nowhere
- Fundamental Theorem is unnecessary
- Why use Riemann sums?
- What is this “Big E”?
- Why use trapezoids?
- Volumes? I thought integrals meant area!
- Integration is opposite of Differentiation; uniqueness of integration is lost
- No use of integrals and derivatives together

What Is Wrong With This Approach?

Confounding the symbolic manipulation of antidifferentiation with the calculation of areas

How I Do Things Now

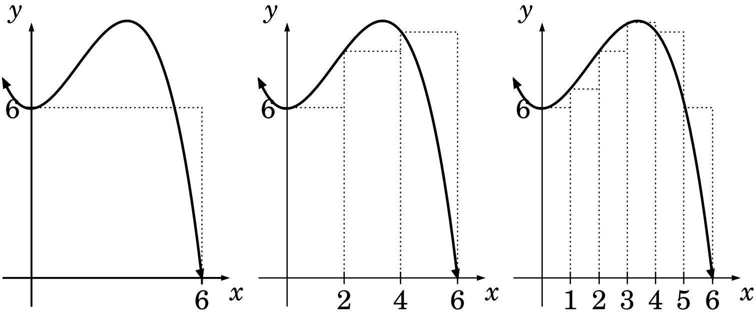
- Riemann Sums
- Trapezoids (and Parabolas)
- Exact Area Formulas
- Antiderivatives
- Fundamental Theorem
- Natural Logarithm/Exponentials
- Hyperbolic Functions
- Integration By Substitution
- Applications (including Volume)
- Differential Equations

What Is Right With This Approach?

- Riemann sums and trapezoids are used to approximate area
- Antiderivatives are used to find exact area
- Fundamental Theorem shows why this works
- Mathematically correct development of the natural logarithm
- “Integral Means Sum”
- Applications combine derivatives and integrals
- “Sum of rates is an amount” – now diff eqs make sense

Outline

Using Rectangles



The area under $f(x) = \frac{1}{6}(36 + 5x^2 - x^3)$.

Using Left, Right, and Midpoint

Problem 1

Consider $y = 8 - x^3$ *over the interval* $[0,2]$.

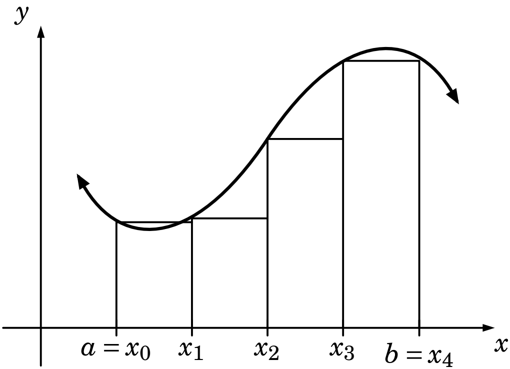
a. *Find the area between this function and the x-axis using a* LRS *with 4 subintervals.*

b. *Find the area between this function and the x-axis using a* RRS *with 4 subintervals.*

c. *Find the area between this function and the x-axis using a* MRS *with 4 subintervals.*

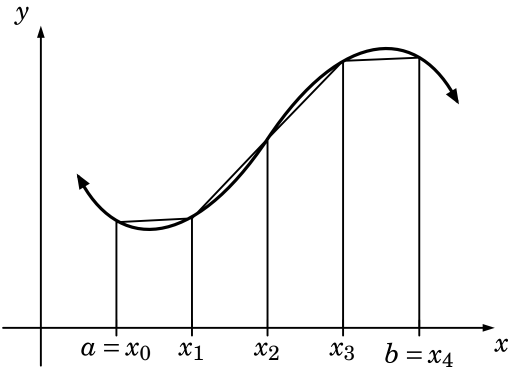
d. *Find the average of the* LRS *and the* RRS. *Is this equal to the* MRS*?*

Rectangles



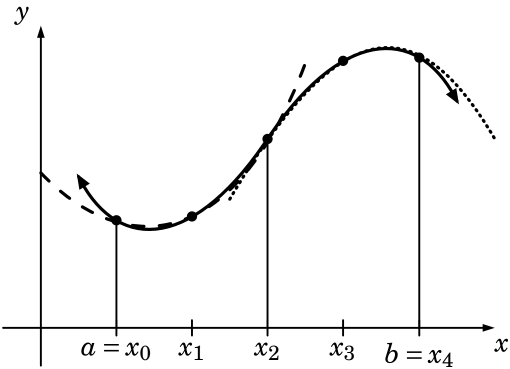
Using rectangles to approximate the area

Trapezoids



Using trapezoids to approximate the area

Parabolas



Using parabolas to approximate the area

Using Trapezoids

Problem 2

Approximate the area under $f(x) = 64 - x^3$ *over the interval* $[0,4]$ *using 4 trapezoids.*

Using Trapezoids

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Approximate the area under $f(x) = 64 - x^3$ *over the interval* $[0,4]$ *using 4 trapezoids.*

Solution.

each subinterval has length $\Delta x = (4-0)/4 = 1$. Then we have

$$\begin{aligned} T_4 &= \frac{\Delta x}{2}[f(0)+2f(1)+2f(2)+2f(3)+f(4)] \\ &= \frac{1}{2}[64+2(63)+2(56)+2(37)+0] = \frac{1}{2}[376] = 184. \end{aligned}$$

Had we used an LRS on 4 subintervals, our approximation would be 220; a RRS gives 156. Note that the trapezoid estimate is the average of the two Riemann sums. ☐

Trapezoid Rule and Simpson's Rule

Problem 3

Consider the region under the curve $y = \frac{1}{x^2+1}$ *and above the x-axis over the interval* $[0, 10]$*. Write out the following sums and evaluate them to find approximations for the area of this region.*

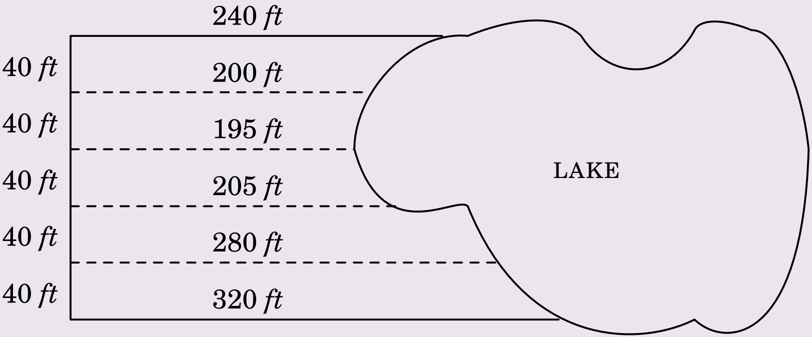
a. *Use the trapezoid rule with 5 equal subintervals.*

b. *Use Simpson's rule with 10 equal subintervals (i.e., five parabolas).*

Application of Trapezoids

Problem 4

Estimate the area of this plot of land.



Application of Trapezoids

Solution.

The trapezoid rule is what we will use to compute this.
The “function” whose area we wish to approximate is given by the distances from the edge of the property to the lakeshore, with $\Delta x = 40$. Hence, the approximate area of the plot of land is

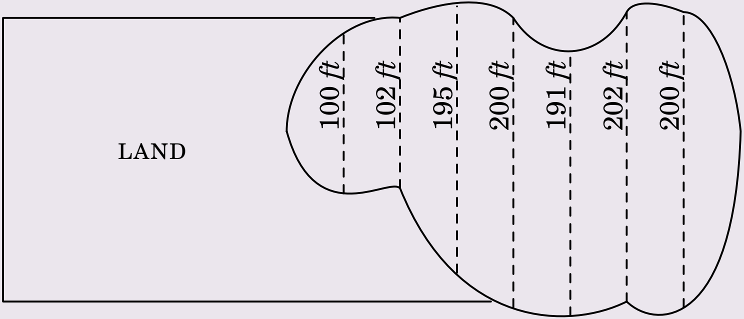
$$\begin{aligned} A &= \frac{40}{2}[240 + 2(200) + 2(195) + 2(205) + 2(280) + 320] \\ &= 20[2320] = 46400 \text{ ft}^2. \end{aligned}$$

☐

Application of Trapezoids

Problem 5

Estimate the surface area of the lake, where the measurements are 36 feet apart.



Application of Trapezoids

Solution.

In this case, using the trapezoid rule with $\Delta x = 36$, we have

$$\begin{aligned} A &= \frac{36}{2}[0 + 2(100) + 2(102) + 2(195) + 2(200) \\ &\qquad + 2(191) + 2(202) + 2(200) + 0] \\ &= 18[2380] = 42840 \text{ ft}^2 \end{aligned}$$

as the area of the lake. ☐

A Tabular Problem

Problem 6

Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	*Leakage (gal / h)*
0	*50*
1	*70*
2	*97*
3	*136*
4	*190*
5	*265*
6	*369*
7	*516*
8	*720*

a. *Find upper and lower estimates of the total quantity of oil that has escaped after 8 hrs.*

b. *The tanker continues to leak 720 gal / hr after the first 8 hours. If the tanker originally contained 25,000 gallons, about how many more hours will elapse in the worst and best cases before all the oil spills?*

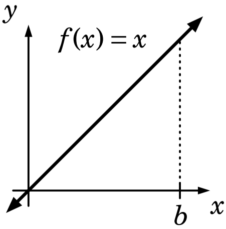
Exact Areas

The exact area under a bounded function $f(x)$ on the interval $[a,b]$ is denoted by

$$\int_a^b f(x)\,dx.$$

The symbol "$\int_a^b$" is called the *definite integral* from a to b.

Exact Area Formulas



The exact area under $f(x) = x$ from 0 to b is that of a right triangle of base b and height b which must be $\frac{1}{2}(b)(b) = \frac{1}{2}b^2$. Hence,

$$\int_0^b x\,dx = \frac{b^2}{2}.$$

Properties of Areas

Theorem 1

Let f and g be bounded continuous functions on the interval $[a,b]$, with $c \in [a,b]$. Let k_1 and k_2 be real constants. Then the following properties hold.

Linearity: $\int_a^b [k_1 f(x) + k_2 g(x)]\ dx = k_1 \int_a^b f(x)\ dx + k_2 \int_a^b g(x)\ dx$

Division of Interval: $\int_a^c f(x)\ dx + \int_c^b f(x)\ dx = \int_a^b f(x)\ dx.$

Reversal of Interval: $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx.$

Comparison: *If $f(x) \geq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

Using Area Formulas

As an example of these properties, consider the *linearity* property. Since areas may be added or subtracted from other areas, we can compute

$$\begin{aligned}\int_0^2 (x^3+5x^2-x)\,dx &= \int_0^2 x^3\,dx + 5\int_0^2 x^2\,dx - \int_0^2 x\,dx\\ &= \frac{2^4}{4} + 5\left(\frac{2^3}{3}\right) - \frac{2^2}{2}\\ &= 4 + 5\left(\tfrac{8}{3}\right) - 2 = \tfrac{52}{3}.\end{aligned}$$

Hence, we have a method to find the area under any polynomial!

Antiderivatives: Keep It Simple

To evaluate the definite integral for $f(x) = x^p$ over $[a, b]$, we write

$$\int_a^b x^p \, dx = \left. \frac{x^{p+1}}{p+1} \right|_a^b = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

Let us find the *derivative* of the expression in the center:

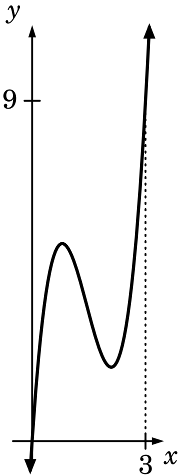
$$\frac{d}{dx}\left(\frac{x^{p+1}}{p+1}\right) = \frac{(p+1)x^{p+1-1}}{p+1} = x^p.$$

The derivative of this expression is the function under which we find the area! In other words,

$$\int x^p \, dx = \frac{x^{p+1}}{p+1}, \qquad \text{and} \qquad \frac{d}{dx}\left(\frac{x^{p+1}}{p+1}\right) = x^p$$

The expression $\frac{x^{p+1}}{p+1}$ is called the *antiderivative*.

Antiderivatives: Keep It Simple



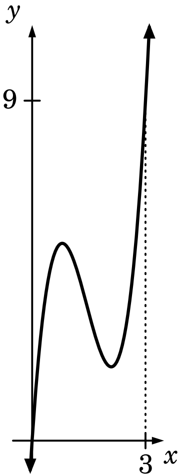
This is the graph of the function $f(x) = 3x^3 - 13x^2 + 15x$. The area under f from 0 to 3 is

$$\int_0^3 (3x^3 - 13x^2 + 15x)\,dx$$

$$= \frac{3x^4}{4} - \frac{13x^3}{3} + \frac{15x^2}{2}\bigg|_0^3$$

$$= \frac{243}{4} - \frac{351}{3} + \frac{135}{2} = \frac{45}{4}.$$

Antiderivatives: Keep It Simple



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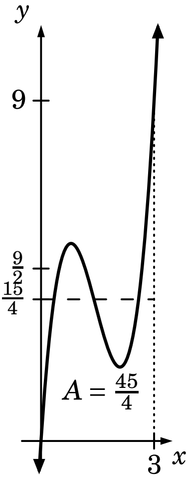
Is it possible to find a rectangle with the same area on the same interval?

The Mean Value Theorem for Integrals



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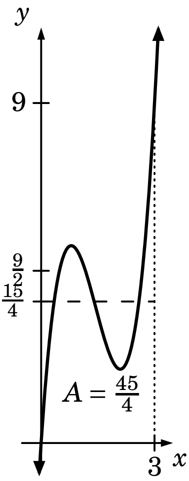
The Mean Value Theorem for Integrals



Is it possible to find a rectangle with the same area on the same interval?

Yes! The area of a rectangle is $A = bh$, where A is given by the definite integral and b is the width of the interval. In this case, $A = \frac{45}{4}$ and $b = 3$. Thus, the correct height of the rectangle with the same area as that under the curve is $h = A \div b = \frac{45}{4} \div 3 = \frac{15}{4}$.

Average Value

Problem 7

Find the average value of the following functions on the interval indicated.

a. $f(x) = x^5 - 2x - 1, \quad [-1, 1]$

b. $g(x) = \cos x, \quad [0, \pi]$

c. $f(x) = 2x^{-3}, \quad [1, 4]$

d. $g(x) = (x-1)^2, \quad [0, 2]$

Application of the Mean Value Theorem

Problem 8

Brayden is caught speeding. The fine is \$3 per minute for each mile per hour above the speed limit. Since he was clocked at speeds as much as 64 mph over a 6-minute period, the judge fines him:

$$(\$3)(\text{no. of minutes})(\text{mph over } 55) = (\$3)(6)(64-55) = \$162$$

Brayden believes the fine is too large since he was going 55 mph at times $t=0$ *and* $t=6$ *minutes, and was going 64 mph only at* $t=3$*. He reckons, in fact, that his speed* v *is given by* $v=55+6t-t^2$*. Brayden argues that since his speed varied, the fine should be determined by calculus rather than by arithmetic. What should he propose to the judge as a reasonable fine?*

Antiderivatives: Keep It Simple

Problem 9

Find antiderivatives of the following. Remember that all differentiation rules can be viewed in reverse as integration rules.

a. $\displaystyle\int \frac{1}{1+x^2}\,dx$

b. $\displaystyle\int \sec^2 x\,dx$

c. $\displaystyle\int 3x^{-4}\,dx$

d. $\displaystyle\int \frac{2}{x^3}\,dx$

e. $\displaystyle\int \sec x \tan x\,dx$

f. $\displaystyle\int -\sin x\,dx$

g. $\displaystyle\int \sin x\,dx$

h. $\displaystyle\int x^{1/3}\,dx$

i. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$

Antiderivatives: Keep It Simple

Problem 10

Using the identity $\tan^2 x + 1 = \sec^2 x$ *to rewrite the integrand first, evaluate* $\int \tan^2 x \, dx$.

Integration By Substitution

Problem 11

Evaluate $\int 3x^2\left(x^3+2\right)^7\,dx.$

Integration By Substitution

Solution.

We make the following *substitution*: Let $u = x^3 + 2$. Then $du = 3x^2\,dx$. Thus, the expression $3x^2\,dx$ becomes simply du.

$$\int 3x^2\left(x^3+2\right)^7\,dx = \int \left(x^3+2\right)^7\,3x^2\,dx = \int u^7\,du = \tfrac{1}{8}u^8 + C.$$

Finally, we “undo” the substitution by replacing every u with $x^3 + 2$, and we obtain

$$\int 3x^2\left(x^3+2\right)^7\,dx = \tfrac{1}{8}\left(x^3+2\right)^8 + C$$

as the antiderivative. ☐

Integration By Substitution

Problem 12

Evaluate $\displaystyle\int \frac{x}{x+1}\,dx.$

Integration By Substitution

Problem 12

Evaluate $\displaystyle\int \frac{x}{x+1}\,dx$.

Solution.

We use the substitution $u = x + 1$. This leads to $du = dx$ and $x = u - 1$. Thus,

$$\int \frac{x}{x+1}\,dx = \int \frac{u-1}{u}\,du = \int \left(1 - \frac{1}{u}\right)\,du$$
$$= u - \ln|u| + C = x + 1 - \ln|x+1| + C$$

is the antiderivative. ☐

Integration By Substitution

Problem 13

a. *Evaluate* $\int \tan x \sec^2 x \, dx$ *using the substitution* $u = \tan x$.

b. *Evaluate* $\int \tan x \sec^2 x \, dx$ *using the substitution* $u = \sec x$.

c. *Explain why your answers to parts (a) and (b), although looking quite different, are actually the same.*

Accumulation Functions

Any function F defined as

$$F(x) = \int_{c}^{x} f(t)\, dt$$

is considered an *accumulation function*. The accumulation function F “accumulates” (or, less formally, “measures”) the area under f from the initial point $t = c$ to the terminal point $t = x$.

Accumulation Functions

Problem 14

If the rate at which water is filling a tank is given by

$$v(t) = t^2 + \frac{1}{1+t^2},$$

where v is measured in gallons per minute, then the amount of water in the tank from $t = 0$ minutes to $t = 1$ minutes is

$$\int_0^1 \left(t^2 + \frac{1}{1+t^2} \right) dt = 1.119 \ \ gallons,$$

rounded to three decimal places.

Accumulation Functions

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rounded to three decimal places.

Integrating a rate gives an amount!

The Fundamental Theorem

Theorem 2 (The Fundamental Theorem of Calculus)

If f is continuous and bounded on the interval $a \le x \le b$, with $c \in [a,b]$, and if F is an antiderivative of f, then

$$\frac{d}{dx}\int_c^x f(t)\,dt = f(x) \tag{1}$$

and

$$\int_c^x f(t)\,dt = F(x) - F(c) \tag{2}$$

The Fundamental Theorem

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and

$$\int_c^x f(t)\,dt = F(x) - F(c) \tag{2}$$

Eq. 1 implies that the rate of change of the area under f is f itself.
Eq. 2 implies that the area under f is found by evaluating its antiderivative.

Accumulation Function Problem

Problem 15

Let F be defined by

$$F(x) = \int_0^x \sqrt{t^2+1}\,dt.$$

a. *Compute $F(0)$.*

b. *Use the trapezoid rule with 4 equal subdivisions to approximate $F(1)$.*

c. *Find the equation of the line tangent to F where $x = 0$.*

Accumulation Function Problem

Solution.

a. $F(0) = \int_0^0 \sqrt{t^2+1}\,dt = 0.$

b. We need to find $F(1) = \int_0^1 \sqrt{t^2+1}\,dt$. This is approximately

$$\begin{aligned} F(1) &\approx \frac{0.25}{2}\Big(\sqrt{0+1} + 2\sqrt{0.25^2+1} + 2\sqrt{0.5^2+1} \\ &\qquad + 2\sqrt{0.75^2+1} + \sqrt{1+1}\Big) \\ &\approx 1.151. \end{aligned}$$

c. The slope is $F'(x) = \sqrt{x^2+1}$, so $F'(0) = 1$. The point is $(0,0)$. Thus the tangent line is $y = x$. ☐

Position, Velocity, Acceleration

Suppose $p(t)$ is position, $v(t)$ is velocity, and $a(t)$ is acceleration. Then

- $p'(t) = v(t)$
- $p''(t) = v'(t) = a(t)$
- $\int a(t)\,dt = v(t) + C$
- $\int v(t)\,dt = p(t) + C$

Teach these together!

Position, Velocity, Acceleration

Suppose $p(t)$ is position, $v(t)$ is velocity, and $a(t)$ is acceleration. Then

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- $p''(t) = v'(t) = a(t)$
- $\int a(t)\,dt = v(t) + C$
- $\int v(t)\,dt = p(t) + C$

Teach these together!

- $\int v(t)\,dt =$ net distance/displacement
- $\int |v(t)|\,dt =$ total distance
- Speed is absolute value of velocity

Position, Velocity, Acceleration

Suppose an object falls from a height of p_0 with initial velocity v_0. In other words, when $t = 0$, we have $p(0) = p_0$ and $v(0) = v_0$. Then we have $a(t) = -g$. We may integrate this with respect to t to obtain

$$v(t) = \int a(t)\,dt = \int -g\,dt = -gt + C_1.$$

Since $v(0) = v_0$, we use this value to compute C_1. So $v(0) = v_0 = -g \cdot 0 + C_1$ implies that $C_1 = v_0$. Hence,

$$v(t) = -gt + v_0.$$

Position, Velocity, Acceleration

Now since $v(t) = p'(t)$, we integrate once more to get

$$p(t) = \int v(t)\,dt = \int (-gt + v_0)\,dt = -\frac{1}{2}gt^2 + v_0 t + C_2.$$

Once more, we use the initial value $p(0) = p_0$ to determine the constant: $p(0) = p_0 = -\frac{1}{2}g \cdot 0 + v_0 \cdot 0 + C_2$ implies $C_2 = p_0$. Finally we have the standard position equation for a falling object,

$$p(t) = -\frac{1}{2}gt^2 + v_0 t + p_0,$$

where g is the acceleration due to gravity, v_0 is the initial velocity, and p_0 is the initial position.

Position, Velocity, Acceleration

Problem 16

A particle moves vertically along the y-axis with velocity given by $v(t) = \exp(\sin t)$ *for* $t \geq 0$*. [Calculator problem]*

a. *In which direction (up or down) is the particle moving at time* $t = 2$*? Why?*

b. *Find the acceleration of the particle at time* $t = 2$*. Is the velocity of the particle increasing at* $t = 2$*?*

c. *Given that* $y(t)$ *is the position of the particle at time t and that* $y(0) = 7$*, find* $y(2)$*.*

d. *Find the total distance traveled by the particle from* $t = 0$ *and* $t = 2$*.*

Position, Velocity, Acceleration

Solution.

a. Since $v(2) = \exp(\sin 2) \approx 2.483 > 0$, the particle is moving up.

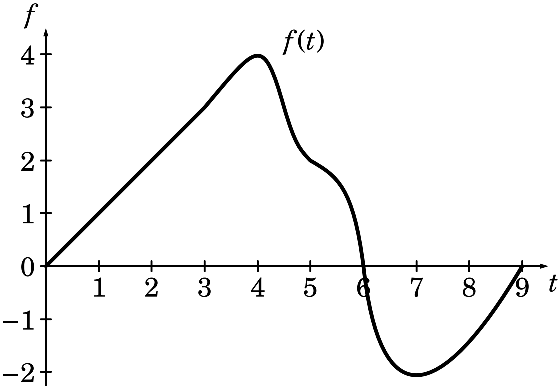
b. Since $v'(2) \approx -1.033 < 0$, the velocity is decreasing.

c. $y(2) = y(0) + \displaystyle\int_0^2 \exp(\sin t)\,dt \approx 7 + 4.237 = 11.237.$

d. The total distance is $\displaystyle\int_0^2 |\exp(\sin t)|\,dt \approx 4.237.$ ☐

Position, Velocity, Acceleration

Suppose f is the differentiable function shown in the figure and that the position at time t seconds of a particle moving along the coordinate axis is $p(t) = \int_0^t f(x)\,dx$ meters.



Position, Velocity, Acceleration

Problem 17

- **a.** *What is the particle's velocity at time $t = 5$?*
- **b.** *Is the acceleration of the particle at time $t = 5$ positive or negative?*
- **c.** *What is the particle's position at $t = 3$?*
- **d.** *At what time during the first 9 seconds does p have its largest value?*
- **e.** *Approximately when is the acceleration zero?*
- **f.** *When is the particle moving toward the origin? Away from the origin?*
- **g.** *On which side of the origin does the particle lie at time $t = 9$?*

Volume by Cross-Sections

If the width across the base of each cross-section is x, and the area of each cross-section is $A(x)$, then the volume is

$$V \approx \Delta x\,(A(x_1)+A(x_2)+\cdots+A(x_n)) = \Delta x \sum_{i=1}^{n} A(x_i).$$

where Δx is the distance between cross-sections. By summing all cross-sections over the entire length of the solid (i.e., by letting $\Delta x \to 0$), we have

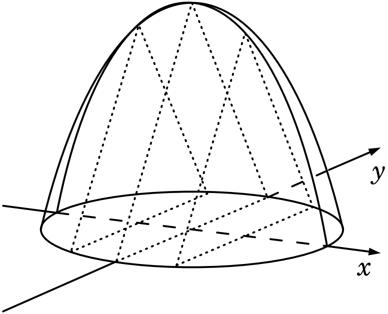
$$V = \int_a^b A(x)\,dx$$

where $b-a$ is the length of the solid and $A(x)$ is the expression for the area of a cross section.

Volume by Cross-Sections

Problem 18

Find the volume of the solid in the figure. The circular base has a radius of 1 and the cross sections perpendicular to the base are equilateral triangles.



Volume by Cross-Sections

Solution.

The distance from the axis to the outer edge of the base is $y = \sqrt{1-x^2}$. Thus, the distance across the circle (from edge to edge) is $2y = 2\sqrt{1-x^2}$. This is also the base of each equilateral triangular cross section. The area of an equilateral triangle with side lenth s is $A(s) = \frac{\sqrt{3}}{4}s^2$. Then

$$A(x) = \frac{\sqrt{3}}{4}(2y)^2 = \frac{\sqrt{3}}{4}\left(2\sqrt{1-x^2}\right)^2 = \sqrt{3}\left(1-x^2\right).$$

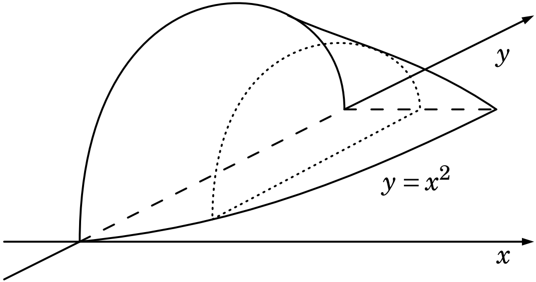
And so

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}\left(1-x^2\right)\,dx = \sqrt{3}\left(x-\tfrac{1}{3}x^3\right)\Big|_{-1}^{1} = \frac{4\sqrt{3}}{3}. \qquad \square$$

Volume by Cross-Sections

Problem 19

Suppose a solid has a base bounded by the line $y = 4$, the curve $y = x^2$, and the y-axis, and whose cross sections are semicircles where the diameters of the semicircles lie in the base. What is the volume?



Volume by Cross-Sections

Solution.

We seek the length of the portion of the cross sections that lie in the base; in this case, that length is the diameter d. The length of each diameter can be expressed by $4-x^2$. Recalling that the area of a semicircle of diameter d is $A(d)=\frac{\pi}{8}d^2$, we have

$$A(x)=\frac{\pi}{8}\left(4-x^2\right)^2=\frac{\pi}{8}(16-8x^2+x^4)=\pi\left(2-x^2+\tfrac{1}{8}x^4\right)$$

for $x\in[0,2]$. Therefore,

$$V=\int_0^2\pi\left(2-x^2+\tfrac{1}{8}x^4\right)\,dx=\pi\left(2x-\tfrac{1}{3}x^3+\tfrac{1}{40}x^5\right)\Big|_0^2=\frac{32\pi}{15}$$

is the volume. ☐

Volume by Cross-Sections

Problem 20

One application is in the X-ray technique of CAT scans. A CAT scan provides a sequence of equally-spaced X-ray images of the cross sections of a patient's organs. The volume of an organ can be approximated by

$$V \approx A(x_1)\Delta x_1 + \cdots + A(x_n)\Delta x_n.$$

Suppose a CAT scan of a human liver shows us X-ray slices spaced 2 cm apart. If the areas of the cross sections are 72, 145, 139, 127, 111, 89, 63, and 22 square centimeters, then estimate the volume of the liver.

Volume by Washers and Disks

Disks Given a region R in the coordinate plane bounded by $f(x)$ and the line $y = k$ over the interval $[a, b]$, then the volume of the solid generated by revolving R about the line $y = k$ is given by

$$V = \pi \int_a^b [f(x) - k]^2 \, dx.$$

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Washers Given a region R in the coordinate plane bounded above by $f(x)$ and below by $g(x)$ over the interval $[a,b]$, then the volume of the solid generated by revolving R about the line $y=k$ is given by

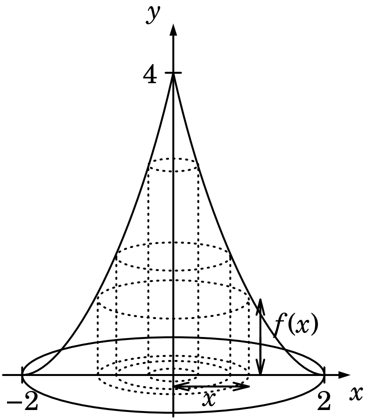
$$V=\pi\int_a^b \left([f(x)-k]^2-[g(x)-k]^2\right)\ dx.$$

Volume By Shells

Shells Given a region R in the coordinate plane bounded by $f(x)$ and the function $g(x)$ over the interval $[a,b]$, then the volume of the solid generated by revolving R about the line $x=k$ is given by

$$V = 2\pi \int_a^b |x-k|(f(x)-g(x))\,dx.$$

Volume By Shells



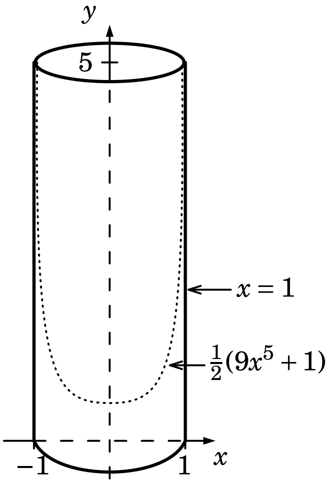
The curve $y = (x-2)^2$ over $[0,2]$ revolved about the y-axis.

Volume By Shells

Problem 21

A drinking glass is modeled by revolving about the y-axis the region R bounded by $f(x) = \frac{1}{2}(9x^5 + 1)$, *the x-axis, the y-axis, and the line* $x = 1$. *If all measurements are in inches, what is the volume of the material needed to construct the glass?*

Volume By Shells



Volume By Shells

Solution.

The interval for R is $[0,1]$. Hence,

$$\begin{aligned} V &= 2\pi \int_0^1 |x-0|(f(x)-0)\,dx = 2\pi \int_0^1 \tfrac{1}{2}x\left(9x^5+1\right)\,dx \\ &= \pi \int_0^1 \left(9x^6+x\right)\,dx = \pi\left(\tfrac{9}{7}x^7+\tfrac{1}{2}x^2\right)\Big|_0^1 \\ &= \pi\left(\tfrac{9}{7}+\tfrac{1}{2}\right) = \frac{25\pi}{14} \end{aligned}$$

☐

Volume By Shells

Problem 22

To make a secondary grip for an umbrella, a manufacturer decides to place a sphere near the base of the umbrella shaft, so that the shaft goes through the sphere. This requires that a sphere of radius 2 cm have a hole of radius 1 cm drilled through it. What is the volume of the resulting spherical ring?

Outline

The Natural Logarithm

Define the function $L(x) = \int_1^x \frac{1}{t}\, dt$.

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- is positive for $x > 1$ and negative for $0 < x < 1$;
- is unbounded so it's range is all real numbers.

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so that $L(kx)$ is also an antiderivative of $\frac{1}{x}$.

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so that $L(kx)$ is also an antiderivative of $\frac{1}{x}$.
Hence, since two antiderivatives can at most differ by a constant, we know that $L(kx) = L(x) + C$. However, when $x = 1$, this becomes $L(k) = L(1) + C$. But we know $L(1) = 0$, so we have $L(k) = C$.
Therefore,

$$L(kx) = L(x) + L(k).$$

The Natural Logarithm

Consider $L(x^p)$ for real p. Then

$$\frac{d}{dx}[L(x^p)] = \frac{1}{x^p} \cdot px^{p-1} = p \cdot \frac{1}{x} = pL'(x).$$

So then we have that $p \cdot \frac{1}{x}$ is the antiderivative of two functions which must only differ by a constant; this gives $L(x^p) = pL(x) + C$.

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So then we have that $p \cdot \frac{1}{x}$ is the antiderivative of two functions which must only differ by a constant; this gives $L(x^p) = pL(x) + C$. Letting $x = 1$ results in $C = 0$. Therefore, in general,

$$L(x^p) = pL(x).$$

The Natural Logarithm

The function L defined by

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

is called the *logarithm* of x.

The Natural Logarithm

What about bases and e?

- Any function f which satisfies the property $f(ab)=f(a)+f(b)$ is of the form $f(x)=cL(x)=c\int_1^x \frac{1}{t}\,dt$ for nonzero constant c.

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$$f(x) = c\log_b(x) = \frac{L(x)}{L(b)}.$$

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- Now, there must be a value of b such that $L(b) = 1$. Call this value e.
- Since b and c are related by $cL(b) = 1$, then when $b = e$, we have $c = 1$. Hence, $L(x) = \log_e(x) = \ln(x)$.

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- Since $\ln(1) = 0$ and $\ln(e) = 1$, then when $x = 0$ and when $x = e$ we get

$$E(0) = 1 \qquad \text{and} \qquad E(1) = e.$$

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- For reals a, b, and p and for positive reals m and n, we let

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- If $b = a$ then $E(a)^2 = E(2a)$.
- Generalize: $E(a)^n = E(na)$ for real n.

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- Taking derivatives of both sides, we get

$$\frac{1}{E(x)} \cdot E'(x) = 1, \qquad \text{or} \qquad E'(x) = E(x).$$

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- Then $E(n) = E(1)^n = e^n$.
- This gives us another way to denote the exponential function $f(x) = \exp(x)$: $f(x) = e^x$.

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One Over-riding Theme

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- Intuitively develop the power rule for antiderivatives as exact area formuals
- Make connections with and through the Fundamental Theorem
- Show what we can do with integrals

Resources

- The MAA's *Resources for Calculus Collection*, five volumes
- The Georgia Association of AP Math Teachers: `http://gaapmt.wikispaces.com`
- College Board: `http://www.collegeboard.com`
- This presentation is housed at my website: `http://www.drchuckgarner.com`

GA^{2}PMT
Georgia Association of Advanced Placement Math Teachers