Scoring Free-Response Questions

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Outline

GAPMT
Georgia Association of Advanced Placement Math Teachers

Part a: 2 points

Students were asked to use the table to estimate $H'(6)$ and then to interpret its meaning in context of the problem with the correct units.

Scoring Guideline Solution:

$$H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

$H'(6)$ is the rate at which the height of the tree is changing, in meters per year, at time $t = 6$.

Part a: Philosophy

First point: Difference quotient with correct estimate of $5/2$. No other estimate earns the point.

Second point: Correct interpretation of context, units, and a moment in time. Three requirements:

1. meaning of H connected to tree;
2. meaning of derivative as a rate of change *with units*;
3. meaning of 6 as time ("at $t = 6$" or "at 6 years").

Part a: Scoring Details

- Units may be with estimate
- Student may refer to tree itself and not height (“tree is increasing/growing/changing ... ”)
- May use estimate in meaning
- We must be convinced that students talk about a rate and NOT an amount: “how fast”, “the speed”, “the rate”, etc.; but “velocity” is not accepted.
- Time as an interval does not earn the point (“during the 6 years”, “over 6 years”, “from 0 to 6 years”)

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Part a: Response 1

$H'(6) = \frac{5}{2}$. The tree is growing at a rate of $5/2$ meters per year at year 6.

Part a: Response 1

$H'(6) = \frac{5}{2}$. The tree is growing at a rate of $5/2$ meters per year at year 6.

Score: 0-1

Part a: Response 2

$$H'(6) = \frac{\int_5^7 H'(t)\,dt}{7-5} = \frac{11-6}{2}$$

The height of the tree is changing at a rate of $\frac{5}{2}$ meters per year.

Part a: Response 2

$$H'(6) = \frac{\int_5^7 H'(t)\,dt}{7-5} = \frac{11-6}{2}$$

The height of the tree is changing at a rate of $\frac{5}{2}$ meters per year.

Score: 1-0

Part a: Response 3

$$H'(6) = \frac{11-6}{7-5} = \frac{5}{2} \text{ meters/year}$$

The rate of change of the tree is $5/2$ at time 6.

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The rate of change of the tree is $5/2$ at time 6.

Score: 1-1

Outline

Part b: 2 points

Students were asked explain why there is at least one time in the interval $2 < t < 10$ such that $H'(t) = 2$.

Scoring Guideline Solution:

$$\frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

Because H is differentiable on $3 \leq t \leq 5$, H is continuous on $3 \leq t \leq 5$. By the MVT, there is a value c, $3 < c < 5$, such that $H'(c) = 2$.

Part b: Philosophy

First point: Correct difference quotient.

Second point: Correct explanation using MVT. Three requirements:

1. connect to our problem;
2. appropriate interval and secant slope;
3. declares continuity

Part b: Scoring Details

- For first point, labeling issues forgiven with correct difference quotient
- For second point, referring to some other theorem or using a formulaic statement does not earn the point.

Part b: Response 1

According to the mean value theorem, where there is an average rate of change over an interval, there must be some location within the interval where the slope is equal to that average rate of change.

Using the points $(5, 6)$ and $(3, 2)$, $H'(t) = \dfrac{6-2}{5-3} = 2$

Therefore there must be at least one time where $H'(t) = 2$.

Part b: Response 1

According to the mean value theorem, where there is an average rate of change over an interval, there must be some location within the interval where the slope is equal to that average rate of change.

Using the points $(5, 6)$ and $(3, 2)$, $H'(t) = \dfrac{6-2}{5-3} = 2$

Therefore there must be at least one time where $H'(t) = 2$.

Score: 1-0

Part b: Response 2

$$\frac{15-2}{10-3}=\frac{13}{7} \qquad \frac{11-2}{7-3}=\frac{9}{4}$$

Because there are secant lines using the data that are on either side of 2, the MVT states that at some point $H'(t)$ must equal 2.

Part b: Response 2

$$\frac{15-2}{10-3}=\frac{13}{7} \qquad \frac{11-2}{7-3}=\frac{9}{4}$$

Because there are secant lines using the data that are on either side of 2, the MVT states that at some point $H'(t)$ must equal 2.

Score: 0-0

Part b: Response 3

According to MVT, $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.
Therefore because $H(t)$ is continuous on the interval $[2,10]$ and differentiable on the open interval $(2,10)$, we can use the MVT to find that:
$H'(t) = \dfrac{f(5)-f(3)}{5-3} = \dfrac{6-2}{2} = 2.$

Part b: Response 3

According to MVT, $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.
Therefore because $H(t)$ is continuous on the interval $[2,10]$ and differentiable on the open interval $(2,10)$, we can use the MVT to find that:
$H'(t) = \dfrac{f(5)-f(3)}{5-3} = \dfrac{6-2}{2} = 2.$

Score: 1-1

Your turn!

Now Score ONLY Parts (a) and (b) from these Samples:
C, F, H

Outline

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Part c: 2 points

Students were asked to approximate the average height of the tree over the interval $2 < t < 10$ using a trapezoidal sum with subintervals as given in the table.

Scoring Guideline Solution:

$$\frac{1}{10-2}\int_2^{10} H(t)\,dt$$
$$\approx \frac{1}{8}\left(\frac{1.5+2}{2}\cdot 1+\frac{2+6}{2}\cdot 2+\frac{6+11}{2}\cdot 2+\frac{11+15}{2}\cdot 3\right)$$
$$=\frac{1}{8}(65.75)=\frac{263}{32}$$

Part c: Philosophy

First point: Correct trapezoidal sum:

1. must have multiplications leading to trapezoidal area, and
2. must have evidence of a sum

Second point: Correct answer.

Part c: Scoring Details

- Maximum 1 error in the trapezoid sum still earns the first point
- For second point, only our answer!
- Linkage errors do not earn second point

Part c: Response 1

$$\frac{3.5}{2} + \frac{8}{2} \cdot 2 + \frac{17}{2} \cdot 2 + \frac{26}{2} \cdot 3$$

Part c: Response 1

$$\frac{3.5}{2} + \frac{8}{2} \cdot 2 + \frac{17}{2} \cdot 2 + \frac{26}{2} \cdot 3$$

Score: 1-0

Part c: Response 2

$$A \approx 1.75 + 8 + 17 + 39$$

$$\text{average height} = \frac{1}{10-2}(A) = \frac{1}{8}(65.75) \text{ meters}$$

Part c: Response 2

$$A \approx 1.75 + 8 + 17 + 39$$

$$\text{average height} = \frac{1}{10-2}(A) = \frac{1}{8}(65.75) \text{ meters}$$

Score: 0-1

Part c: Response 3

$$\text{left} = 1.5 \cdot 1 + 2 \cdot 2 + 6 \cdot 2 + 11 \cdot 4 = 61.5$$
$$\text{right} = 2 \cdot 1 + 6 \cdot 2 + 11 \cdot 2 + 15 \cdot 4 = 96$$
$$\text{Avg} = \frac{1}{8} \cdot \frac{61.5 + 96}{2}$$

Part c: Response 3

$$\text{left} = 1.5 \cdot 1 + 2 \cdot 2 + 6 \cdot 2 + 11 \cdot 4 = 61.5$$
$$\text{right} = 2 \cdot 1 + 6 \cdot 2 + 11 \cdot 2 + 15 \cdot 4 = 96$$
$$\text{Avg} = \frac{1}{8} \cdot \frac{61.5 + 96}{2}$$

Score: 1-0

Your turn!

Now Score ONLY Parts (a) and (b) from these Samples:
C, F, J

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Part d: 3 points

Students were told that the height could be modeled by the function $G(x) = \dfrac{100x}{1+x}$ where x is the diameter of the base of the tree in meters. Students were told that the diameter is increasing at 0.03 meters/year when the tree is 50 meters tall. Finally, students were asked to find the rate of change of the height with respect to time when the tree is 50 meters tall.

Scoring Guideline Solution:

$$\frac{d}{dt}(G(x)) = \frac{d}{dx}(G(x)) \cdot \frac{dx}{dt} = \frac{(1+x)100 - 100x}{(1+x)^2} \cdot \frac{dx}{dt}$$

$$\left.\frac{d}{dt}(G(x))\right|_{x=1} = \frac{100}{4} \cdot 0.03 = \frac{3}{4} \text{ meters per year}$$

Part d: Philosophy

First point: Quotient rule (or product rule)

Second point: Chain rule

Third point: Correct answer with at least second point earned

Part d: Scoring Details

- Correct quotient rule without chain rule loses eligibility for third point.
- Second point earned as long as dG/dx is "quotient-like": looks like an attempt at quotient rule (or product rule) and is a rational function in x.
- We can read with the student for the third point, but only when chain rule point is earned and there is at most one mistake in the quotient rule.
- Notational errors come off the second point, but still eligible for third point if student has our answer

Part d: Response 1

$$\frac{dG}{dt} = \frac{(1+x)100 + 100x}{(1+x)^2} \cdot \frac{dx}{dt}$$
$$\frac{dG}{dt} = \frac{200+100}{4} \cdot 0.03 = \frac{9}{4}$$

Part d: Response 1

$$\frac{dG}{dt} = \frac{(1+x)100 + 100x}{(1+x)^2} \cdot \frac{dx}{dt}$$
$$\frac{dG}{dt} = \frac{200+100}{4} \cdot 0.03 = \frac{9}{4}$$

Score: 0-1-1

Part d: Response 2

$$\frac{dG}{dt} = \frac{(1+x)100 - 100x}{1+x} \cdot \frac{dx}{dt}$$
$$\frac{dG}{dt} = \frac{200-100}{2} \cdot 0.03 = \frac{3}{2}$$

Part d: Response 2

$$\frac{dG}{dt} = \frac{(1+x)100 - 100x}{1+x} \cdot \frac{dx}{dt}$$
$$\frac{dG}{dt} = \frac{200-100}{2} \cdot 0.03 = \frac{3}{2}$$

Score: 0-1-0

Part d: Response 3

$$G'(x) = \frac{(1+x)100 - 100x}{(1+x)^2}\,dx = \frac{(1+1)100 - 100(0.03)}{(1+1)^2} =$$

$$\frac{200-3}{4}$$

$$G'(x) = \frac{197}{4} \text{ m/yr}$$

Part d: Response 3

$$G'(x) = \frac{(1+x)100 - 100x}{(1+x)^2}\,dx = \frac{(1+1)100 - 100(0.03)}{(1+1)^2} = \frac{200-3}{4}$$

$$G'(x) = \frac{197}{4}\ \text{m/yr}$$

Score: 1-0-0

Part d: Response 4

$$g'(x) = \frac{(1+x)(100\frac{dd}{dt}) - (100x)(\frac{dd}{dt})}{(1+x)^2}$$

$$\frac{(1+1)(100 \cdot 0.03) - 100(0.03)}{4} = \frac{2(3)-3}{4}$$

$$g'(x) = \frac{3}{4} \text{ m/yr}$$

Part d: Response 4

$$g'(x) = \frac{(1+x)(100\frac{dd}{dt}) - (100x)(\frac{dd}{dt})}{(1+x)^2}$$

$$\frac{(1+1)(100 \cdot 0.03) - 100(0.03)}{4} = \frac{2(3)-3}{4}$$

$$g'(x) = \frac{3}{4} \text{ m/yr}$$

Score: 1-0-1

Your turn!

Now Score ONLY Parts (a) and (b) from these Samples:
B, F, H

Your turn!

Now Score ALL parts from these Samples:
D, E