# An Approach to Teaching Infinite Series

Chuck Garner, Ph.D.

Department of Mathematics Rockdale Magnet School for Science and Technology

October 11, 2013 / Georgia DOE AP Content Workshop Rockdale Career Academy An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topic

Sequences
Finite Series and Infinit

Geometric Series

Comvenee Tooks

onvergence Tests

Integral Test, p-Series Test

Ratio and Roo

ower Series

Ianipulation of

Taylor Serie

### Outline

Series... Then

Series... Now

**Pre-Series Topics** 

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomials

**Convergence Tests** 

Formal Preliminaries

Integral Test, *p*-Series Test

Comparisons, Alternating Series

Ratio and Root

**Power Series** 

**Manipulation of Series** 

**Taylor Series** 

Summary

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

eries... N

ra Sarias Tonis

Sequences

Finite Series and In Geometric Series

Approximating

onvergence Tests

Formal Preliminaries

Integral Test, p-Series Te Comparisons, Alternatin Series

tio and Root

ver Series

anipulation (

eries

layioi serie

summary

### Outline

Series... Then

Series... Now

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

### **How I Used To Do Things**

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Top

Sequences
Finite Series and

Geometric Series

Convergence Tests

Formal Preliminaries

eries

atio and Root

wer Series

Manipulation of

Tavlor Series

Summary

4 D > 4 P > 4 E > 4 E > E 9 Q C

- Sequences
- ▶ Geometric Series
- Convergence Tests (in one day!: Divergence, Integral, p-Series, Comparisons, Ratio, Root)
- Alternating Series (with absolute/conditional convergence)
- Power Series
- ► Maclaurin Series
- ► Taylor Series

## Things I Noticed Using This Approach

Chuck Garner Series... Then

An Approach to

Teaching Infinite Series

 Confusion between convergence of sequences and convergence of series

- ► Not clear what the "big E" means
- What's the point of Taylor series?
- Why approximate anything don't we have calculators?
- ▶ Why do we need infinite series at all?

### Things I Noticed Using This Approach

Chuck Garner
Series... Then

An Approach to

Teaching Infinite Series

Series... Now

e-Series Topic

- quences nite Series and Infinite ometric Series
- Approximating Polynomial
  - onvergence Tests
- Integral Test, p-Series Test
- Comparisons, Alternating Series

ower Series

Manipulation of Series

Tavlor Series

Summary

 Confusion between convergence of sequences and convergence of series

- ► Not clear what the "big E" means
- ▶ What's the point of Taylor series?
- Why approximate anything don't we have calculators?
- ▶ Why do we need infinite series at all?

Infinite Series feels "tacked on" to the end of the course.

### Outline

Series... Now

**Pre-Series Topics** 

An Approach to Teaching Infinite Series

Chuck Garner

Series... Now

### **How I Do Things Now**

- Use series ideas and concepts whenever possible before teaching series
  - ► *After derivatives*: find limits of sequences
  - Before Riemann sums: evaluate finite series and infinite geometric series; use the divergence test, partial sums
  - ► With applications: approximating polynomials
- Make all the preliminaries explicit
- ► Integral Test and *p*-Series Test
- Comparison Tests
- Alternating Series Test and Error Bound, absolute and conditional convergence
- Ratio and Root Tests
- Power Series
- Manipulation of series
- ► Taylor Series

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

e-Series Topic

Sequences Finite Series and Infinite Geometric Series

pproximum 5 r orynomia

nvergence Tests

Integral Test, p-Series Tes Comparisons, Alternating

atio and Root

wer Series

Manipulation of eries

Taylor Serie

### **How I Do Things Now**

- Use series ideas and concepts whenever possible before teaching series
  - After derivatives: find limits of sequences
  - Before Riemann sums: evaluate finite series and infinite geometric series; use the divergence test, partial sums
  - With applications: approximating polynomials
- Make all the preliminaries explicit
- ► Integral Test and *p*-Series Test
- Comparison Tests
- Alternating Series Test and Error Bound, absolute and conditional convergence
- Ratio and Root Tests
- Power Series
- ► Manipulation of series
- Taylor Series
- ► In red specifically mentioned in the Course Description

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pra-Sariae Toni

Sequences
Finite Series and Infinit

Approximating Polyno

onvergence Tests

Formal Preliminaries
Integral Test, p-Series Test

eries

ower Series

lanipulation o

laylor Serie

### What Is Right With This Approach?

- ▶ Builds on previous knowledge
- Provides a rationale for wanting to know about convergence
- ► Taylor series is the point of knowing series!
- ► Takes one more day, but the understanding is so much better

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topics

Sequences

Geometric Series

Convergence Tests

ivergence Tests

Integral Test, p-Series Test Comparisons, Alternating

atio and Root

. .

wer Series

anipulation of

avlor Series

### Outline

Series... Now

### **Pre-Series Topics**

An Approach to Teaching Infinite Series

Chuck Garner

#### Pre-Series Topics

### **Sequences**

I teach sequences after teaching derivatives and l'Hôpital's Rule.

- Approach sequences as an application of l'Hôpital's Rule and infinite limits
- ► Define convergence and divergence
- ► Introduce the idea of *domination*: for example, *n*<sup>n</sup> dominates *n*!

 $n^n \gg n!$ 

An Approach to Teaching Infinite Series

Chuck Garner

Series... The

Series...

Pre-Series Top

#### Sequences

Geometric Series

ripproximiting rolynomia

#### ivergence Tests

Integral Test, p-Series Tes Comparisons, Alternating

Datio and Doot

. .

ower Series

Manipulation of

avlor Series

### Sequences

An Approach to Teaching Infinite Series

Chuck Garner

#### Sequences

I teach sequences after teaching derivatives and l'Hôpital's Rule.

- ► Approach sequences as an application of l'Hôpital's Rule and infinite limits
- Define convergence and divergence
- ▶ Introduce the idea of domination: for example,  $n^n$ dominates n!

 $n^n \gg n!$ 

Typical question: Does  $\sqrt[n]{n!}$  dominate  $\arctan n$ ?

### **Finite Series**

I teach finite series before teaching integration/Riemann sums.

- Compute with summation notation
- ▶ Determine formulas for  $\sum n, \sum n^2, \sum n^3$
- Determine sums of arithmetic and geometric series

An Approach to Teaching Infinite Series

Chuck Garner

Finite Series and Infinite Geometric Series

### **Geometric Series**

I teach geometric series right after finite series.

- Partial sums; notion of series convergence
- ▶ If the series  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \to 0$  as  $n \to \infty$ . (This is the contrapositive of the divergence test.)
- ► Show harmonic series diverges
- ► Infinite geometric series
- ► Applications of geometric series

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... N

Pre-Series Topic

Finite Series and Infinite Geometric Series

approximating Polynomials

Convergence Tests

Integral Test, p-Series Test
Comparisons, Alternating

Ratio and Root

wer Series

wer beries

Series

Taylor Serie

### **Geometric Series**

An Approach to Teaching Infinite Series

Chuck Garner

Series... Ther

series... Now

Sequences
Finite Series and Infinite

Geometric Series

Approximating Polynomials

Convergence Tests

Formal Preliminaries
Integral Test, p-Series Test

eries

tatio and Root

wer Series

(animulation

eries

Taylor Serie

Summary

I teach geometric series right after finite series.

- ► Partial sums; notion of series convergence
- ► If the series  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \to 0$  as  $n \to \infty$ . (This is the contrapositive of the divergence test.)
- ► Show harmonic series diverges
- ► Infinite geometric series
- Applications of geometric series

Typical question: The series

$$\sum_{n=0}^{\infty} 3(2x-1)^n$$

is geometric. What values of *x* lead to convergence?

I teach approximating polynomials with applications of the derivative.

The tangent line is a linear approximation L(x) to a function f(x) (also called the linearization).

#### Problem 1

*Use a linear approximation to estimate*  $\sqrt{77}$ .

An Approach to Teaching Infinite Series

Chuck Garner

Approximating Polynomials

I teach approximating polynomials with applications of the derivative.

The tangent line is a linear approximation L(x) to a function f(x) (also called the linearization).

#### Problem 1

Use a linear approximation to estimate  $\sqrt{77}$ .

### Solution.

The tangent line to  $f(x) = \sqrt{x}$  centered at x = 81 is

$$L(x) = f(81) + f'(81)(x - 81) = 9 + \frac{1}{2\sqrt{81}}(x - 81).$$

Then 
$$f(77) \approx L(77) = 9 + \frac{1}{18}(77 - 81) = 9 - \frac{2}{9} = 8\frac{7}{9}$$
.

A linear approximation to f matches the slope of f. A quadratic approximation to f should match both the slope and the concavity of f. So we assume the approximation has the form

$$Q(x) = L(x) + C(x - a)^2$$

where *a* is the center. Then Q''(x) = 2C. Since we want Q''(a) = f''(a), we find that  $C = \frac{1}{2}f''(a)$ . Then

$$Q(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^{2}.$$

An Approach to Teaching Infinite Series

Chuck Garner

Series... Ther

Series...

Pre-Series Topi

Sequences

Geometric Series

#### Approximating Polynomials

nvergence Tests

Integral Test, p-Series Test Comparisons, Alternating

atio and Root

ower Series

Manipulation of Series

aylor Series

By the same logic, a cubic approximator should match the third derivative of f. So we assume the approximation has the form

$$B(x) = L(x) + Q(x) + C(x - a)^3$$

where *a* is the center. Then B'''(x) = 6C. Since we want B'''(a) = f'''(a), we find that  $C = \frac{1}{6}f'''(a)$ . Then

$$B(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f'''(a)}{6}(x - a)^3.$$

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Finite Series and Infinite

Approximating Polynomials

nvergence Tests

formal Preliminaries

ntegral Test, p-Series Test

io and Root

Power Serie

Power Series

Manipulation of Series

Taylor Serie

### **Outline**

Series... Then

Series... Now

**Pre-Series Topics** 

Sequences

Finite Series and Infinite Geometric Serie
Approximating Polynomials

### **Convergence Tests**

Formal Preliminaries

Integral Test, p-Series Test

Comparisons, Alternating Series

Ratio and Root

**Power Series** 

**Manipulation of Series** 

**Taylor Serie** 

Summar

#### An Approach to Teaching Infinite Series

Chuck Garner

eries... Then

eries... Now

re-Series Topic .

Finite Series and Infini Geometric Series

Approximating Polynomial

#### Convergence Tests

Formal Preliminaries

omparisons, Alternati

Ratio and Root

yyor Corios

wer Series

Ianipulation of eries

aylor Series

### **Preliminaries**

### Explicitly define:

- ► Partial Sums
- Divergence Test
- ▶ Geometric Series Test

#### An Approach to Teaching Infinite Series

Chuck Garner

Formal Preliminaries

### **Integral Test**

#### An Approach to Teaching Infinite Series

Chuck Garner

Integral Test, p-Series Test

### Theorem 1 (The Integral Test)

Suppose f is a positive, decreasing, continuous function on the interval  $[1,\infty)$  and let  $a_n = f(n)$ . Then if  $\int_1^\infty f(x) dx$  converges (diverges), then  $\sum_{n=1}^{\infty} a_n$  converges (diverges).

Use the Integral Test to prove (again) that the harmonic series diverges.

### **Integral Test**

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Sequences

Geometric Series

Convergence Test

Integral Test, p-Series Test

Comparisons, Alternating Series

ower Series

Manipulation of Series

Taylor Series

Summary

Theorem 2 (Remainder Estimate for the Integral Test)

If the infinite series  $\sum a_n$  converges by the Integral Test, then it has sum S and

$$\int_{k+1}^{\infty} f(x) \, dx \le R_k \le \int_{k}^{\infty} f(x) \, dx$$

where  $R_k = S - s_k$  is the remainder and  $f(n) = a_n$ .

### **Problem 2**

What is the error bound in using  $\sum_{n=1}^{5} \frac{1}{n^2}$  to estimate the value of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ?

What is the error bound in using  $\sum_{n=1}^{5} \frac{1}{n^2}$  to estimate the value of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ?

Solution.

The estimate has an error bound

$$\int_6^\infty \frac{1}{x^2} dx \le R_5 \le \int_5^\infty \frac{1}{x^2} dx,$$

or

$$\frac{1}{6} \le R_5 \le \frac{1}{5}.$$

Hence, the difference between the exact infinite sum *S* and the estimate  $\sum_{n=1}^{5} \frac{1}{n^2} \approx 1.4636$  is in the interval [0.167, 0.2]. (The remainder is, in fact, about 0.1813.)

Series... Then

Series... Now

Pre-Series Topi

Finite Series and In

Geometric Series

onvergence Test

Integral Test, p-Series Test
Comparisons, Alternating

Ratio and Root

oruge Carios

wer Series

Manipulation of Series

Taylor Series

### *p*-Series Test

Theorem 3 (*p*-Series Test)

The infinite series  $\sum 1/n^p$ , where p is a positive real number, converges if p > 1 and diverges if  $p \le 1$ .

An Approach to Teaching Infinite Series

Chuck Garner

Integral Test, p-Series Test

### **Integral Test**

#### **Problem 4**

How many terms of the series  $\sum_{n=1}^{\infty} \frac{1}{n^5}$  are needed so that the error is less than 0.01?

#### Solution.

we require

$$R_k \le \int_k^\infty \frac{1}{x^5} \, dx < 0.01.$$

Hence, we compute

$$\int_{k}^{\infty} \frac{1}{x^5} dx = \lim_{b \to \infty} \left( -\frac{1}{4b^4} + \frac{1}{4k^4} \right) = \frac{1}{4k^4} < 0.01 = \frac{1}{100}.$$

So  $100 < 4k^4$  implies  $25 < k^4$ . Thus we need k = 3 (or more) terms.

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Nov

Pre-Series Topi

Sequences Finite Series and Inf

Geometric Series Approximating Polynomi

onvergence Tests

Formal Preliminaries
Integral Test, p-Series Test

mparisons, Alternating ries

unio una 1001

Power Series

Manipulation of Series

Taylor Series

### **Direct Comparison and Limit Comparison**

Comparison Tests are based on dominance.

Theorem 4 (The Direct Comparison Test)

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be positive series and suppose  $b_n \gg a_n$ . Then if  $\sum_{n=1}^{\infty} b_n$  converges, so does  $\sum_{n=1}^{\infty} a_n$ .

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

re-Series Topi

Sequences Finite Series and Infinit

Approximating Polyno

Convergence Tests

Formal Preliminaries
Integral Test, p-Series Test
Comparisons, Alternating
Series

Ratio and Root

ower Series

Manipulation of

Tavlor Serie

### **Direct Comparison and Limit Comparison**

#### An Approach to Teaching Infinite Series

Chuck Garner

Comparisons, Alternating Series

### Theorem 5 (The Limit Comparison Test)

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two series such that  $a_n \ge 0$  and  $b_n > 0$ . Let

$$\lim_{n\to\infty}\frac{a_n}{b_n}=L.$$

Then we have the following three cases.

- 1. If  $0 < L < \infty$ , then  $\sum a_n$  and  $\sum b_n$  either both converge or both diverge.
- **2.** If  $L = \infty$  and  $\sum a_n$  converges, then  $\sum b_n$  also converges.
- **3.** If L = 0 and  $\sum a_n$  diverges, then  $\sum b_n$  also diverges.

Formal Preliminaries

Comparisons, Alternating Series

Ratio and Root

ower Series

Manipulation of

Taylor Serie

Summary

4 D > 4 A > 4 E > 4 E > 5 P 9 Q P

How do you decide what to use to compare? Check the limit as  $n \to \infty$  to get a *p*-Series. Since

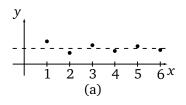
$$\frac{n}{\sqrt{n^5 + n^3 - 1}} \to \frac{n}{n^{5/2}} \to \frac{1}{n^{3/2}}$$

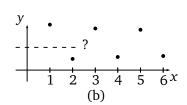
we see that

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5 + n^3 - 1}} \le \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Thus,  $\sum n/\sqrt{n^5+n^3-1}$  converges since  $\sum 1/n^{3/2}$  converges.

## **Alternating Series**





- (a) Partial sums of  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$
- (b) Partial sums of  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$

#### An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Sequences

Geometric Series

spproximating rotynomia:

vergence Tests

Integral Test, p-Series Test
Comparisons, Alternating
Series

Ratio and Root

ower Series

Manipulation of Series

Taylor Serie

### **Alternating Series**

Theorem 6 (The Alternating Series Test) The alternating series

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if all the following are satisfied.

- **1.** All terms of  $a_n$  are positive.
- **2.** The sequence  $\{a_n\}$  is decreasing.
- **3.** The sequence  $\{a_n\}$  approaches zero as  $n \to \infty$ .

An Approach to Teaching Infinite Series

Chuck Garner

Comparisons, Alternating Series

### **Alternating Series**

# Theorem 6 (The Alternating Series Test)

The alternating series

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges if all the following are satisfied.

- **1.** All terms of  $a_n$  are positive.
- **2.** The sequence  $\{a_n\}$  is decreasing.
- **3.** The sequence  $\{a_n\}$  approaches zero as  $n \to \infty$ .

### **Theorem 7 (Alternating Series Error Bound)**

Let  $\sum (-1)^n a_n$  be a convergent alternating series. The error in using the kth partial sum  $s_k$  to estimate the sum S is less than the absolute value of the (k+1)th term of the sequence  $a_{k+1}$ .

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Sequences Finite Series and Infinit

Approximating Pol

Convergence Tests

Formal Preliminaries

Comparisons, Alternating Series

atio and Root

wer Series

wer Series

Manipulation of Series

aylor Series

### **Absolutely Convergent**

Simple definition of absolutely convergent: If  $\sum |a_n|$  converges, so does  $\sum a_n$ . Conditionally convergent means that  $\sum |a_n|$  converges, while  $\sum a_n$  does not.

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Sequences
Finite Series and Infin

Approximating Polynomials

nvergence Tests

Integral Test, p-Series Test Comparisons, Alternating Series

Ratio and Root

ower Series

Manipulation of Series

Tavlor Serie

### **Absolutely Convergent**

Simple definition of absolutely convergent:

If  $\sum |a_n|$  converges, so does  $\sum a_n$ .

Conditionally convergent means that  $\sum |a_n|$  converges, while  $\sum a_n$  does not.

#### Problem 5

We wish to determine the absolute convergence, conditional convergence, or divergence of

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^5}$$

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Finite Series and Infini Geometric Series

approximating rotynomials

nvergence Tests

Integral Test, p-Series Test Comparisons, Alternating Series

Ratio and Root

ower Series

Manipulation of

avlor Series

### **Absolutely Convergent**

Simple definition of absolutely convergent:

If  $\sum |a_n|$  converges, so does  $\sum a_n$ .

Conditionally convergent means that  $\sum |a_n|$  converges, while  $\sum a_n$  does not.

#### Problem 5

We wish to determine the absolute convergence, conditional convergence, or divergence of

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^5}$$

Solution.

Since  $\left| \frac{\sin n}{n^5} \right| < \frac{1}{n^5}$  and  $\sum \frac{1}{n^5}$  is a convergent *p*-Series, the series  $\sum \left| \frac{\sin n}{n^5} \right|$  converges absolutely.

An Approach to Teaching Infinite Series

Chuck Garner

Comparisons, Alternating Series

#### **Ratio Test**

In a geometric series, the ratio of consecutive terms is r: For  $\sum ar^n$ ,

$$\frac{ar^{k+1}}{ar^k} = r$$

The Ratio Test relies on something similar: For  $\sum a_n$ , we check

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}.$$

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Tonic

Sequences

Geometric Series

Convencence Tests

onvergence Tests

Integral Test, p-Series Test Comparisons, Alternating

Ratio and Root

ower Series

Manipulation of Series

Tavlor Serie

### **Ratio Test**

In a geometric series, the ratio of consecutive terms is r: For  $\sum ar^n$ ,

$$\frac{ar^{k+1}}{ar^k} = r.$$

The Ratio Test relies on something similar: For  $\sum a_n$ , we check

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k}.$$

### Theorem 8 (The Ratio Test)

For the series  $\sum_{n=1}^{\infty} a_n$ , define

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If r < 1, then  $\sum a_n$  converges absolutely; if r > 1, then  $\sum a_n$  diverges.

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Finite Series and Infinite

Approximating Polynomial

nvergence Tests

Integral Test, p-Series Tes Comparisons, Alternating

Ratio and Root

ower Series

Power Series

Series

Taylor Serie

Summary

4□ → 4回 → 4 亘 → 4 亘 → 9 0 0 0

## **Root Test**

#### Theorem 9 (The Root Test)

For the series  $\sum_{n=1}^{\infty} a_n$ , define

$$r = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

If r < 1, then  $\sum a_n$  converges absolutely; if r > 1, then  $\sum a_n$  diverges.

#### An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

BCIICS... INOW

Pre-Series Topi

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomials

nvergence Tests

Integral Test, *p*-Series Test Comparisons, Alternating Series

Ratio and Root

wer Series

Manipulation of

avlor Series

Summarv

### **Root Test**

## Theorem 9 (The Root Test)

For the series  $\sum_{n=1}^{\infty} a_n$ , define

$$r = \lim_{n \to \infty} \sqrt[n]{|a_n|}.$$

If r < 1, then  $\sum a_n$  converges absolutely; if r > 1, then  $\sum a_n$  diverges.

Any series in which you could ue the Ratio Test, you can also use the Root Test – as long as you know  $\sqrt[n]{n!} \to \infty$  as  $n \to \infty$ . (Use the Root Test for geometric series too!)

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Tonio

Sequences

Finite Series and Infinite Geometric Series

anvence of Tests

ivergence Tests

Integral Test, p-Series Test Comparisons, Alternating Series

#### Ratio and Root

Power Serie

Manipulation of Series

Taylor Series

## **Outline**

Series... Then

Series... Now

**Pre-Series Topics** 

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomials

**Convergence Tests** 

Formal Preliminaries

Integral Test, p-Series Test

Comparisons, Alternating Series

Ratio and Roo

#### **Power Series**

**Manipulation of Series** 

**Taylor Serie** 

Summar

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

e-Series Topic

Sequences

Geometric Series

Approximating Polynomials

vergence Tests

Formal Preliminaries
Integral Test, p-Series Test
Comparisons, Alternating
Series

Ratio and l

Power Series

Manipulation of Series

aylor Series

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

series... Now

ra Sarias Tanio

Sequences
Finite Series and Info

Geometric Series

Convergence Tests

Formal Preliminaries

Integral Test, p-Series Test Comparisons, Alternating

io and Root

Power Series

ower belies

Manipulation of Series

Taylor Serie

Summary

Power series are functions of x.

- ► Basic power series is geometric
- Geometric series give our intuitive notion of an interval of convergence
- ► Give the students three examples...

#### Problem 6

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

#### Solution.

We apply the Root Test:

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{(x-1)^n}{n!} \right|} = |x-1| \lim_{n \to \infty} \left| \frac{1}{\sqrt[n]{n!}} \right| = |x-1| \cdot 0 = 0.$$

This limit will always be zero no matter the value of x, we see that this power series converges absolutely for all real numbers x.

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topic

Sequences
Finite Series and Infin

Geometric Series

Convergence Tests

ormal Preliminaries ntegral Test, p-Series Test omparisons, Alternating

atio and Root

Power Series

#### Power Series

Manipulation of Series

Taylor Serie

#### Problem 7

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} n!(x+2)^n$$

Solution.

By the Root Test, we have

$$\lim_{n \to \infty} \sqrt[n]{|n!(x+2)^n|} = |x+2| \lim_{n \to \infty} \sqrt[n]{|n!|} = \infty.$$

Since this limit is infinite, we see that this power series diverges for all real numbers, except for the trivial value x = -2. So, as opposed to converging for all real numbers in the previous example, this series converges only for a single value.

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

eries... l

re-Series Topi

Sequences Finite Series and I

Approximating D

Convergence Tests

nal Preliminaries

egral Test, p-Series Te: nparisons, Alternatin; ies

io and Root

Power Series

Manipulation of Series

Taylor Serie

Summary

#### Problem 8

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$$

#### Solution.

We use the Root Test.

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{(x-4)^n}{n+1} \right|} = |x-4| \lim_{n \to \infty} \sqrt[n]{\left| \frac{1}{n+1} \right|} = |x-4| \cdot 1 = |x-4|$$

This series will converge if the limit is less than 1; hence, we must have |x-4| < 1 for convergence. Thus,

-1 < x - 4 < 1, or 3 < x < 5, are the values of x that result in a convergent series.

However, the last problem is incomplete. The Root Test is inconclusive if the limit is equal to 1. So what happens if x = 3 or x = 5?

- ▶ If x = 3, the power series becomes  $\sum_{n=0}^{\infty} (-1)^n / (n+1)$ which converges by the Alternating Series Test.
- ▶ If x = 5, the power series becomes  $\sum_{n=0}^{\infty} 1/(n+1)$  which diverges by the Integral Test.

Therefore, the values of x which result in  $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$  being a convergent series are in the interval  $3 \le x < 5$ .

# **Radius of Convergence**

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

re-Series Topic

Sequences Finite Series and Infinite

Approximating Polynomials

onvergence Tests

Integral Test, p-Series Test Comparisons, Alternating Series

Lomparisons, Aiternat Series Ratio and Root

Power Series

Manipulation of Series

Taylor Serie

Summary

## Theorem 10 (Radius of Convergence of Power Series) Let $\sum a_n(x-a)^n$ be a power series. If

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \quad or \quad \lim_{n \to \infty} \sqrt[n]{|a_n|} = r$$

is a positive real number, then  $\frac{1}{r}$  is the radius of convergence of the power series; if r=0, then  $\infty$  is the radius of convergence; and if  $r=\infty$ , then 0 is the radius of convergence.

## **Outline**

Series... Then

Series... Now

**Pre-Series Topics** 

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomials

**Convergence Tests** 

Formal Preliminaries

Integral Test, p-Series Test

Comparisons, Alternating Series

Ratio and Roo

**Power Series** 

**Manipulation of Series** 

**Taylor Serie** 

Summar

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

eries... Now

o Conice Toni

re-Series Topic Seguences

Finite Series and Infinite Geometric Series

approximating Polynomials

vergence Tests

Formal Preliminaries Integral Test, *p*-Series Test Comparisons, Alternating Series

atio and Root

ower Series

Manipulation of Series

Taylor Series

ummary

4 D > 4 图 > 4 E > 4 E > 9 Q ()

# **Manipulation of Series**

From the geometric series  $\sum x^n = \frac{1}{1-x}$ , we may get – on the interval of convergence – power series for

- ►  $\frac{1}{1-x^2}$  by replacing  $x^2$  for x in  $\frac{1}{1-x}$
- ▶  $\frac{1}{1+x}$  by replacing -x for x in  $\frac{1}{1-x}$
- ►  $\frac{1}{1+x^2}$  by replacing  $-x^2$  for x in  $\frac{1}{1-x}$
- $ightharpoonup \frac{1}{(1-x)^2}$  by differentiating  $\frac{1}{1-x}$
- ►  $-\ln|1-x|$  by integrating  $\frac{1}{1-x}$
- arctan x by integrating  $\frac{1}{1+x^2}$
- ▶  $\ln |1 + x|$  by integrating  $\frac{1}{1+x}$
- ▶  $\frac{x^2}{(1+x)^3}$  by differentiating  $\frac{1}{1+x}$  twice and then multiplying by  $x^2$ .

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Topi

Sequences Finite Series and

Geometric Series

Convergence Tests

ormal Preliminaries

Integral Test, p-Series Test Comparisons, Alternating

atio and Root

. .

wer Series

Manipulation of Series

Taylor Serie

# **Manipulation of Series**

From the geometric series  $\sum x^n = \frac{1}{1-x}$ , we may get – on the interval of convergence – power series for

- ▶  $\frac{1}{1-x^2}$  by replacing  $x^2$  for x in  $\frac{1}{1-x}$
- ▶  $\frac{1}{1+x}$  by replacing -x for x in  $\frac{1}{1-x}$
- ▶  $\frac{1}{1+x^2}$  by replacing  $-x^2$  for x in  $\frac{1}{1-x}$
- $ightharpoonup \frac{1}{(1-x)^2}$  by differentiating  $\frac{1}{1-x}$
- ►  $-\ln|1-x|$  by integrating  $\frac{1}{1-x}$
- arctan x by integrating  $\frac{1}{1+x^2}$
- ▶  $\ln |1 + x|$  by integrating  $\frac{1}{1+x}$
- $\frac{x^2}{(1+x)^3}$  by differentiating  $\frac{1}{1+x}$  twice and then multiplying by  $x^2$ .

Check endpoints!

An Approach to Teaching Infinite Series

Chuck Garner

eries... Then

Series... Now

Pre-Series Topi

Sequences Finite Series and :

Geometric Series

Convergence Tests

Formal Preliminaries

Integral Test, p-Series Test Comparisons, Alternating Series

atio and Root

wer Series

Manipulation of

Series

Taylor Serie

## Outline

Series... Now

Taylor Series

An Approach to Teaching Infinite Series

Chuck Garner

**Taylor Series** 

### Taylor Series is the finale!

- Reintroduce the "approximating polynomial" idea as the first few terms of a power series
- ► Generate more terms of power series through the notion that  $f^{(n)}(a) = n!C$
- ► In this way, the *idea* of Taylor series is simply a continuation of what came before

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series...

. . . . . .

Sequences

Geometric Series

Convergence Tests

nvergence Tests

Integral Test, p-Series Test Comparisons, Alternating

atio and Root

ower Series

ower Series

Manipulation of Series

Taylor Series

# Taylor Series is the finale!

- Reintroduce the "approximating polynomial" idea as the first few terms of a power series
- ► Generate more terms of power series through the notion that  $f^{(n)}(a) = n!C$
- ► In this way, the *idea* of Taylor series is simply a continuation of what came before
- ▶ Now put it all together: Taylor series constitute
  - approximating polynomials
  - questions of convergence
  - power series and intervals of convergence
  - differentiation
  - manipulation of series

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series...

Pre-Series Top

Sequences Finite Series and Ir

Approximating Polynon

nvergence Tests

Integral Test, p-Series Test Comparisons, Alternating

atio and Root

wer Series

Manipulation of

Taylor Series

## Theorem 11 (Taylor's Theorem)

Let f be a function such that  $f^{(k+1)}(x)$  exists for all x in the interval (a-r,a+r). Then

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the kth degree Taylor polynomial of f at a, and

$$R_k(x) \le \frac{|f^{(k+1)}(c)|}{(k+1)!} |x-a|^{k+1}$$

is the Lagrange form of the remainder, where c is a number between a and x which maximizes  $f^{(k+1)}$ . Moreover, assume f has derivatives of all orders. Then  $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$  if and only if  $R_k \to 0$  as  $k \to \infty$ .

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

eries... Now

re-Series Topics

Finite Series and Infinite Geometric Series

vergence Tests

Integral Test, p-Series Test Comparisons, Alternating Series

Ratio and Re

Power Series

Manipulation of Series

Taylor Series

#### **Problem 9**

Use the third-order Taylor polynomial for  $f(x) = \ln x$  centered at x = 1 to approximate  $\ln 1.06$ .

#### Solution.

The Taylor polynomial is

$$P_3(x) = x - 1 - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$$

so the approximation is

$$P_3(1.06) = 0.06 - \frac{0.06^2}{2} + \frac{0.06^3}{3}$$
  
= 0.06 - 0.0018 + 0.000072 = 0.058272.  $\square$ 

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... No

Dra-Sariae Ton

Sequences Finite Series and Inf

Geometric Series

Convergence Tests

mal Preliminaries

Integral Test, p-Series Test Comparisons, Alternating Series

Ratio and Root

war Sories

ver series

fanipulation of eries

Taylor Series

An Approach to Teaching Infinite Series

Chuck Garner

**Taylor Series** 

Note that the Lagrange form of the remainder,

$$R_k(x) \le \frac{|f^{(k+1)}(c)|}{(k+1)!} |x-a|^{k+1},$$

is another remainder which is simply the next term of the series.

▶ Major difference: we maximize the numerator on the interval [a,x] to get the largest bound.

#### Problem 10

Determine the error in using the third-order Taylor polynomial for  $f(x) = \ln x$  centered at x = 1 to approximate  $\ln 1.06$ .

#### Solution.

The remainder is

$$R_3(x) \le \frac{|f^{(4)}(c)|}{4!}|x-1|^4.$$

The fourth derviative of  $\ln x$  is  $-6/x^4$  whose maximum value on the interval [1, 1.06] is when x = 1; this leads to a numerator of -6. So the error must be less than

$$R_3(1.06) \le \frac{|-6|}{4!} |0.06|^4 = \frac{0.00001296}{4} = 0.00000324. \quad \Box$$

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Top

Finite Series and Infin Geometric Series

Approximating Polyn

nvergence Tests

Integral Test, p-Series Tes Comparisons, Alternating

atio and Root

Power Series

Manipulation of Series

**Taylor Series** 

## **Outline**

Series... Then

Series... Now

**Pre-Series Topics** 

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomials

**Convergence Tests** 

Formal Preliminaries

Integral Test, p-Series Test

Comparisons, Alternating Series

Ratio and Root

**Power Series** 

**Manipulation of Series** 

**Taylor Series** 

**Summary** 

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

eries... Now

\_\_\_\_

e-Series Topic

inite Series and Inf Geometric Series

Approximating Polynomials

vergence Tests

Formal Preliminaries
integral Test, p-Series Test

eries

. .

wer Series

anipulation c eries

aylor Series

Summary

•

Don't wait for the end of the year to introduce series concepts!

Start as early as possible

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Pre-Series Tonio

Sequences
Finite Series and Infini

Geometric Series

nvergence Tests

Formal Preliminaries Integral Test, p-Series Test

io and Root

.............

wer Series

Manipulation of Series

aylor Series

Don't wait for the end of the year to introduce series concepts!

- ► Start as early as possible
- ► Talk about convergence early even with limits

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

D 0 ' m '

Sequences

Finite Series and Infinit Geometric Series

ripproximating rotynomia.

nvergence Tests

Formal Preliminaries Integral Test, p-Series Test

ies

atio and Root

wer Series

ver Series

Ianipulation of eries

Taylor Serie

Don't wait for the end of the year to introduce series concepts!

- ► Start as early as possible
- ► Talk about convergence early even with limits
- Use more than the tangent line to approximate functions

An Approach to Teaching Infinite Series

Chuck Garner

Series... Then

Series... Now

Sequences

Finite Series and Infinite Geometric Series

Approximating Polynomian

nvergence Test

Integral Test, p-Series Test Comparisons, Alternating

tatio and Root

ver Series

wer Series

Ianipulation of eries

avlor Series

Don't wait for the end of the year to introduce series concepts!

- Start as early as possible
- ► Talk about convergence early even with limits
- Use more than the tangent line to approximate functions
- ► Taylor series and the Lagrange remainder should be easy to learn and a fitting conclusion

An Approach to Teaching Infinite Series

Chuck Garner

### Resources

- An Approach to Teaching Infinite Series
- Chuck Garner

- Summary

- ► The MAA's Resources for Calculus Collection, five volumes
- ► The Georgia Association of AP Math Teachers: http://gaapmt.wikispaces.com
- ► College Board: http://www.collegeboard.com
- ► This presentation is housed at my website: http://www.drchuckgarner.com

