An Approach to Teaching Infinite Series

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Outline

Series... Then
Series... Now
Pre-Series Topics
Sequences
Finite Series and Infinite Geometric Series
Approximating Polynomials
Convergence Tests
Formal Preliminaries
Integral Test, p-Series Test
Comparisons, Alternating Series
Ratio and Root
Power Series
Manipulation of Series
Taylor Series
Summary

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Summary

How I Used To Do Things

- Sequences
- Geometric Series
- Convergence Tests (in one day!: Divergence, Integral, p-Series, Comparisons, Ratio, Root)
- Alternating Series (with absolute/conditional convergence)
- Power Series
- Maclaurin Series
- Taylor Series

Things I Noticed Using This Approach

- Confusion between convergence of sequences and convergence of series
- Not clear what the “big E” means
- What’s the point of Taylor series?
- Why approximate anything – don’t we have calculators?
- Why do we need infinite series at all?

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- Why do we need infinite series at all?

Infinite Series feels "tacked on" to the end of the course.

Outline

How I Do Things Now

- Use series ideas and concepts whenever possible *before* teaching series
  - *After derivatives:* find limits of sequences
  - *Before Riemann sums:* evaluate finite series and infinite geometric series; use the divergence test, partial sums
  - *With applications:* approximating polynomials
- Make all the preliminaries explicit
- Integral Test and p-Series Test
- Comparison Tests
- Alternating Series Test and Error Bound, absolute and conditional convergence
- Ratio and Root Tests
- Power Series
- Manipulation of series
- Taylor Series

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- Manipulation of series
- Taylor Series
- In red specifically mentioned in the Course Description

What Is Right With This Approach?

- Builds on previous knowledge
- Provides a rationale for wanting to know about convergence
- Taylor series is the point of knowing series!
- Takes one more day, but the understanding is so much better

Outline

Sequences

I teach sequences after teaching derivatives and l'Hôpital's Rule.

- Approach sequences as an application of l'Hôpital's Rule and infinite limits
- Define convergence and divergence
- Introduce the idea of *domination*: for example, n^n dominates $n!$

$$n^n \gg n!$$

Sequences

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- Introduce the idea of *domination*: for example, n^n dominates $n!$

$$n^n \gg n!$$

Typical question: Does $\sqrt[n]{n!}$ dominate $\arctan n$?

Finite Series

I teach finite series before teaching integration/Riemann sums.

- Compute with summation notation
- Determine formulas for $\sum n$, $\sum n^2$, $\sum n^3$
- Determine sums of arithmetic and geometric series

Geometric Series

I teach geometric series right after finite series.

- Partial sums; notion of series convergence
- If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$. (This is the contrapositive of the divergence test.)
- Show harmonic series diverges
- Infinite geometric series
- Applications of geometric series

Geometric Series

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- Applications of geometric series

Typical question: The series

$$\sum_{n=0}^{\infty} 3(2x-1)^n$$

is geometric. What values of x lead to convergence?

Approximating Polynomials

I teach approximating polynomials with applications of the derivative.

The tangent line is a linear approximation $L(x)$ to a function $f(x)$ (also called the *linearization*).

Problem 1

Use a linear approximation to estimate $\sqrt{77}$.

Approximating Polynomials

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Problem 1

Use a linear approximation to estimate $\sqrt{77}$.

Solution.

The tangent line to $f(x) = \sqrt{x}$ centered at $x = 81$ is

$$L(x) = f(81) + f'(81)(x - 81) = 9 + \frac{1}{2\sqrt{81}}(x - 81).$$

Then $f(77) \approx L(77) = 9 + \frac{1}{18}(77 - 81) = 9 - \frac{2}{9} = 8\frac{7}{9}$. □

Approximating Polynomials

A linear approximation to f matches the slope of f. A quadratic approximation to f should match both the slope and the concavity of f. So we assume the approximation has the form

$$Q(x) = L(x) + C(x-a)^2$$

where a is the center. Then $Q''(x) = 2C$. Since we want $Q''(a) = f''(a)$, we find that $C = \frac{1}{2}f''(a)$. Then

$$Q(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2.$$

Approximating Polynomials

By the same logic, a cubic approximator should match the third derivative of f. So we assume the approximation has the form

$$B(x) = L(x) + Q(x) + C(x-a)^3$$

where a is the center. Then $B'''(x) = 6C$. Since we want $B'''(a) = f'''(a)$, we find that $C = \frac{1}{6}f'''(a)$. Then

$$B(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3.$$

Outline

Preliminaries

Explicitly define:

- Partial Sums
- Divergence Test
- Geometric Series Test

Integral Test

Theorem 1 (The Integral Test)

Suppose f is a positive, decreasing, continuous function on the interval $[1, \infty)$ and let $a_n = f(n)$. Then if $\int_1^\infty f(x)\,dx$ converges (diverges), then $\sum_{n=1}^\infty a_n$ converges (diverges).

Use the Integral Test to prove (again) that the harmonic series diverges.

Integral Test

Theorem 2 (Remainder Estimate for the Integral Test)

If the infinite series $\sum a_n$ *converges by the Integral Test, then it has sum* S *and*

$$\int_{k+1}^{\infty} f(x)\,dx \leq R_k \leq \int_{k}^{\infty} f(x)\,dx$$

where $R_k = S - s_k$ *is the remainder and* $f(n) = a_n$.

Problem 2

What is the error bound in using $\sum_{n=1}^{5} \frac{1}{n^2}$ *to estimate the value of* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *?*

Integral Test

Problem 3

What is the error bound in using $\sum_{n=1}^{5} \frac{1}{n^2}$ *to estimate the value of* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *?*

Solution.

The estimate has an error bound

$$\int_6^\infty \frac{1}{x^2}\,dx \leq R_5 \leq \int_5^\infty \frac{1}{x^2}\,dx,$$

or

$$\frac{1}{6} \leq R_5 \leq \frac{1}{5}.$$

Hence, the difference between the exact infinite sum S and the estimate $\sum_{n=1}^{5} \frac{1}{n^2} \approx 1.4636$ is in the interval $[0.167, 0.2]$. (The remainder is, in fact, about 0.1813.) □

p-Series Test

Theorem 3 (p-Series Test)

The infinite series $\sum 1/n^p$, *where* p *is a positive real number, converges if* $p > 1$ *and diverges if* $p \leq 1$.

Integral Test

Problem 4

How many terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ are needed so that the error is less than 0.01?

Solution.

we require

$$R_k \leq \int_k^{\infty} \frac{1}{x^5}\,dx < 0.01.$$

Hence, we compute

$$\int_k^{\infty} \frac{1}{x^5}\,dx = \lim_{b\to\infty}\left(-\frac{1}{4b^4} + \frac{1}{4k^4}\right) = \frac{1}{4k^4} < 0.01 = \frac{1}{100}.$$

So $100 < 4k^4$ implies $25 < k^4$. Thus we need $k = 3$ (or more) terms. □

Direct Comparison and Limit Comparison

Comparison Tests are based on *dominance*.

Theorem 4 (The Direct Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive series and suppose $b_n \gg a_n$. Then if $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

Direct Comparison and Limit Comparison

Theorem 5 (The Limit Comparison Test)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series such that $a_n \geq 0$ and $b_n > 0$. Let

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then we have the following three cases.

1. *If $0 < L < \infty$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.*
2. *If $L = \infty$ and $\sum a_n$ converges, then $\sum b_n$ also converges.*
3. *If $L = 0$ and $\sum a_n$ diverges, then $\sum b_n$ also diverges.*

Direct Comparison and Limit Comparison

How do you decide what to use to compare? Check the limit as $n \to \infty$ to get a p-Series. Since

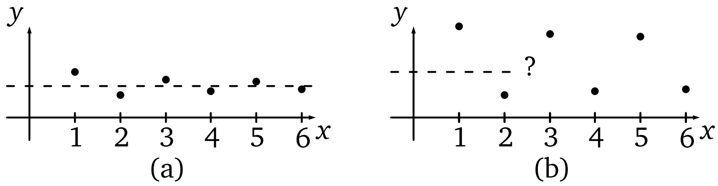
$$\frac{n}{\sqrt{n^5+n^3-1}} \to \frac{n}{n^{5/2}} \to \frac{1}{n^{3/2}}$$

we see that

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n^3-1}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}.$$

Thus, $\sum n/\sqrt{n^5+n^3-1}$ converges since $\sum 1/n^{3/2}$ converges.

Alternating Series



(a) Partial sums of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

(b) Partial sums of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$

Alternating Series

Theorem 6 (The Alternating Series Test)

The alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n$$

converges if all the following are satisfied.

1. *All terms of a_n are positive.*
2. *The sequence $\{a_n\}$ is decreasing.*
3. *The sequence $\{a_n\}$ approaches zero as $n \to \infty$.*

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Theorem 7 (Alternating Series Error Bound)

Let $\sum(-1)^n a_n$ be a convergent alternating series. The error in using the kth partial sum s_k to estimate the sum S is less than the absolute value of the $(k+1)$th term of the sequence a_{k+1}.

Absolutely Convergent

Simple definition of *absolutely convergent*:
If $\sum |a_n|$ converges, so does $\sum a_n$.
Conditionally convergent means that $\sum |a_n|$ converges, while $\sum a_n$ does not.

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Problem 5

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Solution.
Since $\left|\frac{\sin n}{n^5}\right| < \frac{1}{n^5}$ and $\sum \frac{1}{n^5}$ is a convergent p-Series, the series $\sum \left|\frac{\sin n}{n^5}\right|$ converges absolutely. □

Ratio Test

In a geometric series, the ratio of consecutive terms is r: For $\sum ar^n$,

$$\frac{ar^{k+1}}{ar^k} = r.$$

The Ratio Test relies on something similar: For $\sum a_n$, we check

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k}.$$

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Theorem 8 (The Ratio Test)

For the series $\sum_{n=1}^{\infty} a_n$, define

$$r = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

If $r < 1$, then $\sum a_n$ converges absolutely; if $r > 1$, then $\sum a_n$ diverges.

Root Test

Theorem 9 (The Root Test)

For the series $\sum_{n=1}^{\infty} a_n$*, define*

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $r < 1$*, then* $\sum a_n$ *converges absolutely; if* $r > 1$*, then* $\sum a_n$ *diverges.*

Root Test

Theorem 9 (The Root Test)

For the series $\sum_{n=1}^{\infty} a_n$, *define*

$$r = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $r < 1$, *then* $\sum a_n$ *converges absolutely; if* $r > 1$, *then* $\sum a_n$ *diverges.*

Any series in which you could ue the Ratio Test, you can also use the Root Test – as long as you know $\sqrt[n]{n!} \to \infty$ as $n \to \infty$. (Use the Root Test for geometric series too!)

Outline

Power Series

Power series are functions of x.

- Basic power series is geometric
- Geometric series give our intuitive notion of an *interval of convergence*
- Give the students three examples...

Power Series

Problem 6

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

Solution.

We apply the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-1)^n}{n!}\right|} = |x-1| \lim_{n\to\infty} \left|\frac{1}{\sqrt[n]{n!}}\right| = |x-1| \cdot 0 = 0.$$

This limit will always be zero no matter the value of x, we see that this power series converges absolutely for all real numbers x. □

Power Series

Problem 7

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} n!(x+2)^n$$

Solution.

By the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{|n!(x+2)^n|} = |x+2| \lim_{n\to\infty} \sqrt[n]{|n!|} = \infty.$$

Since this limit is infinite, we see that this power series diverges for all real numbers, except for the trivial value $x = -2$. So, as opposed to converging for all real numbers in the previous example, this series converges only for a single value. □

Power Series

Problem 8

Determine values of x for which the power series converges.

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$$

Solution.

We use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^n}{n+1}\right|} = |x-4| \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n+1}\right|} = |x-4|\cdot 1 = |x-4|$$

This series will converge if the limit is less than 1; hence, we must have $|x-4| < 1$ for convergence. Thus, $-1 < x-4 < 1$, or $3 < x < 5$, are the values of x that result in a convergent series. □

Interval of Convergence

However, the last problem is incomplete. The Root Test is inconclusive if the limit is equal to 1. So what happens if $x = 3$ or $x = 5$?

- If $x = 3$, the power series becomes $\sum(-1)^n/(n+1)$ which converges by the Alternating Series Test.
- If $x = 5$, the power series becomes $\sum 1/(n+1)$ which diverges by the Integral Test.

Therefore, the values of x which result in $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n+1}$ being a convergent series are in the interval $3 \leq x < 5$.

Radius of Convergence

Theorem 10 (Radius of Convergence of Power Series)
Let $\sum a_n(x-a)^n$ be a power series. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r \quad \text{or} \quad \lim_{n\to\infty}\sqrt[n]{|a_n|} = r$$

is a positive real number, then $\frac{1}{r}$ is the radius of convergence of the power series; if $r = 0$, then ∞ is the radius of convergence; and if $r = \infty$, then 0 is the radius of convergence.

Outline

Manipulation of Series

From the geometric series $\sum x^n = \frac{1}{1-x}$, we may get – on the interval of convergence – power series for

- $\frac{1}{1-x^2}$ by replacing x^2 for x in $\frac{1}{1-x}$
- $\frac{1}{1+x}$ by replacing $-x$ for x in $\frac{1}{1-x}$
- $\frac{1}{1+x^2}$ by replacing $-x^2$ for x in $\frac{1}{1-x}$
- $\frac{1}{(1-x)^2}$ by differentiating $\frac{1}{1-x}$
- $-\ln|1-x|$ by integrating $\frac{1}{1-x}$
- $\arctan x$ by integrating $\frac{1}{1+x^2}$
- $\ln|1+x|$ by integrating $\frac{1}{1+x}$
- $\frac{x^2}{(1+x)^3}$ by differentiating $\frac{1}{1+x}$ twice and then multiplying by x^2.

Manipulation of Series

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Check endpoints!

Outline

Taylor Series

Taylor Series is the finale!

- Reintroduce the "approximating polynomial" idea as the first few terms of a power series
- Generate more terms of power series through the notion that $f^{(n)}(a) = n!C$
- In this way, the *idea* of Taylor series is simply a continuation of what came before

Taylor Series

Taylor Series is the finale!

- Reintroduce the "approximating polynomial" idea as the first few terms of a power series
- Generate more terms of power series through the notion that $f^{(n)}(a) = n!C$
- In this way, the *idea* of Taylor series is simply a continuation of what came before
- Now put it all together: Taylor series constitute
  - approximating polynomials
  - questions of convergence
  - power series and intervals of convergence
  - differentiation
  - manipulation of series

Taylor Series

Theorem 11 (Taylor's Theorem)

Let f be a function such that $f^{(k+1)}(x)$ exists for all x in the interval $(a-r, a+r)$. Then

$$P_k(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the kth degree Taylor polynomial of f at a, and

$$R_k(x) \leq \frac{|f^{(k+1)}(c)|}{(k+1)!}|x-a|^{k+1}$$

is the Lagrange form of the remainder, where c is a number between a and x which maximizes $f^{(k+1)}$. Moreover, assume f has derivatives of all orders. Then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ if and only if $R_k \to 0$ as $k \to \infty$.

Taylor Series

Problem 9

Use the third-order Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$ to approximate $\ln 1.06$.

Solution.

The Taylor polynomial is

$$P_3(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

so the approximation is

$$\begin{aligned} P_3(1.06) &= 0.06 - \frac{0.06^2}{2} + \frac{0.06^3}{3} \\ &= 0.06 - 0.0018 + 0.000072 = 0.058272. \quad \square \end{aligned}$$

Taylor Series

Note that the Lagrange form of the remainder,

$$R_k(x) \leq \frac{|f^{(k+1)}(c)|}{(k+1)!}|x-a|^{k+1},$$

is another remainder which is simply the next term of the series.

- Major difference: we maximize the numerator on the interval $[a,x]$ to get the largest bound.

Taylor Series

Problem 10

Determine the error in using the third-order Taylor polynomial for $f(x) = \ln x$ *centered at* $x = 1$ *to approximate* $\ln 1.06$.

Solution.

The remainder is

$$R_3(x) \leq \frac{|f^{(4)}(c)|}{4!}|x-1|^4.$$

The fourth derviative of $\ln x$ is $-6/x^4$ whose maximum value on the interval $[1, 1.06]$ is when $x = 1$; this leads to a numerator of -6. So the error must be less than

$$R_3(1.06) \leq \frac{|-6|}{4!}|0.06|^4 = \frac{0.00001296}{4} = 0.00000324. \quad \square$$

Outline

One Over-riding Theme

Don't wait for the end of the year to introduce series concepts!

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One Over-riding Theme

Don't wait for the end of the year to introduce series concepts!

- Start as early as possible
- Talk about convergence early – even with limits
- Use more than the tangent line to approximate functions
- Taylor series and the Lagrange remainder should be easy to learn and a fitting conclusion

Resources

- The MAA's *Resources for Calculus Collection*, five volumes
- The Georgia Association of AP Math Teachers: `http://gaapmt.wikispaces.com`
- College Board: `http://www.collegeboard.com`
- This presentation is housed at my website: `http://www.drchuckgarner.com`

GA²PMT
Georgia Association of Advanced Placement Math Teachers