The Gambler, the Stutterer, and the Cubic
An Historical Account of the Solution of the Cubic

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Outline

1. The Controversy
2. The Cubic Solution
3. Lives of Tartaglia and Cardano
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1. The Controversy
2. The Cubic Solution
3. Lives of Tartaglia and Cardano
It all started in 1515...
Succeeded in solving cubics of the form $x^3 + mx = n$

Did not publish this discovery, but told his student, Antonio Fior

But in 1535 someone else claimed to have solved cubics of the form $x^3 + px^2 = q...$
Fior challenged Tartaglia to a public contest...
The Controversy

Dispute!

- Fior challenged Tartaglia to a public contest…
- …which Tartaglia won!
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…which Tartaglia won!
Origins

NICOLO FONTANA (TARTAGLIA)
1499-1557

“However, by good fortune, only eight days before the time fixed for collecting the two sets of thirty sealed problems, I had discovered the general rule for such expressions.”
Dispute!

- Fior challenged Tartaglia to a public contest…
- …which Tartaglia won!
- But it gets worse: Enter Cardano
Girolamo Cardano
1501-1576

“Those arts which are, to be sure, not finite, as geometry and arithmetic, do not suffer adornment; others, contrarily, are rather subject to division and embellishment, such as astronomy and jurisprudence.”
Dispute!

- Published *Ars Magna* in 1545; included Tartaglia’s cubic solutions
The Controversy

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- ...Tartaglia lost!
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- Another public challenge was set…
- …Tartaglia lost!
- Ferrari rose to fame and wealth; Tartaglia was broke
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Cardano’s Method

We solve $x^3 + 6x = 20$. 

Let $u^3 - v^3 = 20$ and $uv = 2$. Then $x^3 + (3uv)x = u^3 - v^3$. Since $(a - b)^3 + 3ab(a - b) = a^3 - b^3$, we see that $x = u - v$ solves the equation.

Now we determine $u$ and $v$:

$u^3 = 20 + v^3 = 20 + 2u^3$

which can be written as $u^6 - 20u^3 - 8 = 0$ with solutions $u^3 = 10 \pm \sqrt{108}$. Hence, $v^3 = -10 \pm \sqrt{108}$.

Finally, $x = u - v = 3\sqrt[3]{10} \pm \sqrt[3]{108}$.
Cardano’s Method

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- Let \( u^3 - v^3 = 20 \) and \( uv = 2 \). Then

\[
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- Since \((a - b)^3 + 3ab(a - b) = a^3 - b^3\), we see that \( x = u - v \) solves the equation.
- Now we determine \( u \) and \( v \):
  \[ u^3 = 20 + v^3 = 20 + \left( \frac{2}{u} \right)^3 = 20 + \frac{8}{u^3}, \]
  which can be written as
  \[ u^6 - 20u^3 - 8 = 0 \]
  with solutions \( u^3 = 10 \pm \sqrt{108} \). Hence, \( v^3 = -10 \pm \sqrt{108} \).
Cardano’s Method

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- Let $u^3 - v^3 = 20$ and $uv = 2$. Then
  
  $$x^3 + (3uv)x = u^3 - v^3.$$  

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- Now we determine $u$ and $v$:
  
  $$u^3 = 20 + v^3 = 20 + \left( \frac{2}{u} \right)^3 = 20 + \frac{8}{u^3}$$

  which can be written as
  
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  with solutions $u^3 = 10 \pm \sqrt{108}$. Hence, $v^3 = -10 \pm \sqrt{108}$.

- Finally, $x = u - v = \sqrt[3]{10} + \sqrt[3]{108} - \sqrt[3]{-10} + \sqrt[3]{108}$. 

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Cardano’s Method

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Cardano’s Method

- However, we see that $x = 2$ solves $x^3 + 6x = 20$.
- So $\sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}} = 2$

General solution of $x^3 + mx = n$ is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$
Cardano’s Method

The solution of \( ax^3 + bx^2 + cx + d = 0 \) can be found by converting it to Cardano’s cubic. Using \( x = z - \frac{b}{3a} \), we change variables:

\[
\begin{align*}
\quad & a \left( z - \frac{b}{3a} \right)^3 + b \left( z - \frac{b}{3a} \right)^2 + c \left( z - \frac{b}{3a} \right) + d = 0 \\
& a \left( z^3 - \frac{3b}{3a} z^2 + \frac{3b^2}{9a^2} z - \frac{b^3}{27a^3} \right) + b \left( z^2 - \frac{2b}{3a} z + \frac{b^2}{9a^2} \right) + cz - \frac{bc}{3a} + d = 0 \\
& az^3 + \left( c - \frac{b^2}{3a} \right) z + \frac{2b^3}{27a^2} - \frac{bc}{3a} + d = 0 \\
& z^3 + \left( \frac{c}{a} - \frac{b^2}{3a^2} \right) z + \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = 0
\end{align*}
\]

Dividing through by \( a \) puts this in the form \( z^3 + mz = n \).
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3. Lives of Tartaglia and Cardano
Lives of Tartaglia and Cardano

Tartaglia

- Born Nicolo Fontana in 1499
- Obtained a split skull and a sabre cut to the face in 1512
- Applied mathematics to artillery fire (directly contradicting Aristotle)

Cardano

- Published editions of Euclid and Archimedes
- Wrote one of the best 16th century math books
- Well-regarded in his lifetime
- Taught science and math all his life; died in 1557
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Cardano

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- Primary career as a physician
- Also a gambler, mathematician, astrologer, teacher
- Imprisoned for casting a horoscope of Jesus
- But later became astrologer to the Pope
- Cardano was the first to accept negative and imaginary numbers as legitimate solutions
- Died in 1576 of suicide