The Gambler, the Stutterer, and the Cubic

An Historical Account of the Solution of the Cubic

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Outline

1. The Controversy
2. The Cubic Solution
3. Lives of Tartaglia and Cardano

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2. The Cubic Solution

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Origins

It all started in 1515…

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SCIPIONE DEL FERRO
1465-1526

- Succeeded in solving cubics of the form $x^3 + mx = n$
- Did not publish this discovery, but told his student, Antonio Fior
- But in 1535 someone else claimed to have solved cubics of the form $x^3 + px^2 = q \ldots$

Dispute!

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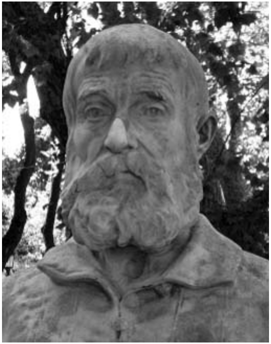

NICOLO FONTANA (TARTAGLIA)

1499-1557

"However, by good fortune, only eight days before the time fixed for collecting the two sets of thirty sealed problems, I had discovered the general rule for such expressions."

Dispute!

- Fior challenged Tartaglia to a public contest…
- …which Tartaglia won!
- But it gets worse: Enter Cardano

Cardano

GIROLAMO CARDANO
1501-1576

"Those arts which are, to be sure, not finite, as geometry and arithmetic, do not suffer adornment; others, contrarily, are rather subject to division and embellishment, such as astronomy and jurisprudence."

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- …Tartaglia lost!
- Ferrari rose to fame and wealth; Tartaglia was broke

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- Now we determine u and v:

$$u^3 = 20 + v^3 = 20 + \left(\frac{2}{u}\right)^3 = 20 + \frac{8}{u^3}$$

which can be written as

$$u^6 - 20u^3 - 8 = 0$$

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  with solutions $u^3 = 10 \pm \sqrt{108}$. Hence, $v^3 = -10 \pm \sqrt{108}$
- Finally, $x = u - v = \sqrt[3]{10 + \sqrt{108}} - \sqrt[3]{-10 + \sqrt{108}}$.

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General solution of $x^3 + mx = n$ is

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}} - \sqrt[3]{-\frac{n}{2} + \sqrt{\left(\frac{n}{2}\right)^2 + \left(\frac{m}{3}\right)^3}}.$$

Cardano's Method

The solution of $ax^3 + bx^2 + cx + d = 0$ can be found by converting it to Cardano's cubic. Using $x = z - \frac{b}{3a}$, we change variables:

$$
\begin{aligned}
a\left(z - \tfrac{b}{3a}\right)^3 + b\left(z - \tfrac{b}{3a}\right)^2 + c\left(z - \tfrac{b}{3a}\right) + d &= 0 \\
a\left(z^3 - \tfrac{3b}{3a}z^2 + \tfrac{3b^2}{9a^2}z - \tfrac{b^3}{27a^3}\right) + b\left(z^2 - \tfrac{2b}{3a}z + \tfrac{b^2}{9a^2}\right) + cz - \tfrac{bc}{3a} + d &= 0 \\
az^3 + \left(c - \tfrac{b^2}{3a}\right)z + \tfrac{2b^3}{27a^2} - \tfrac{bc}{3a} + d &= 0 \\
z^3 + \left(\tfrac{c}{a} - \tfrac{b^2}{3a^2}\right)z + \tfrac{2b^3}{27a^3} - \tfrac{bc}{3a^2} + \tfrac{d}{a} &= 0
\end{aligned}
$$

Dividing through by a puts this in the form $z^3 + mz = n$.

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- Born Nicolo Fontana in 1499
- Obtained a split skull and a sabre cut to the face in 1512
- Applied mathematics to artillery fire (directly contradicting Aristotle)

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- Obtained a split skull and a sabre cut to the face in 1512
- Applied mathematics to artillery fire (directly contradicting Aristotle)
- Published editions of Euclid and Archimedes
- Wrote one of the best 16th century math books
- Well-regarded in his lifetime
- Taught science and math all his life; died in 1557

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Cardano

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Cardano

- Born in 1501
- Primary career as a physician
- Also a gambler, mathematican, astrologer, teacher
- Imprisoned for casting a horoscope of Jesus
- But later became astrologer to the Pope
- Cardano was the first to accept negative and imaginary numbers as legitimate solutions
- Died in 1576 of suicide