What I Learned Teaching AP Calculus

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Georgia Association of Advanced Placement Math Teachers

Outline

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How I Started

What I Learned

Questions

How I Started Teaching AP Calculus

I taught students the way I was taught...

- Algebra/Trig review
- Limits
- Derivatives
- Applications of Derivatives
- Integrals
- Applications of Integrals
- BC-only topics

Well What's Wrong With That?

The traditional sequence wasn't working for my students.

- Was it me?
- Was it them?
- Was it the course sequence?
- Was it their preparation?
- Was it my preparation?
- Was it the book?
- Was it the activities?
- Was it all of the above?

How Do I Fix It?

I had to really understand

- why calculus is taught the way it is,
- the knowledge my students were bringing to the class,
- why AP includes the topics it does,
- why AP does not include certain topics, and
- calculus itself.

Outline

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Here are some of the things I learned.

Keep the Algebra Minimized

Compute the following.

1. $\lim_{x\to 1} \dfrac{3x^3+7x^2-2x-8}{(x-1)^5}$
2. $\lim_{x\to 3} \dfrac{\sqrt{x+3}-\sqrt{6}}{x-3}$
3. $\dfrac{d}{dx}\left(\dfrac{(2x-3)^5(3x-2)^4}{(x-4)^3}\right)$

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Are those more instructive than the following?

1. $\lim_{x\to 3} \frac{x^2-9}{x-3}$
2. $\lim_{x\to 1} \frac{(x-1)^2}{x-1}$
3. $\frac{d}{dx}\left(\frac{(x-1)(x-2)}{x}\right)$

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3. definitions of continuity and the derivative;
4. l'Hôpital's rule.

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Limits mean behavior, and we report behavior with y-values.

Limits As Behavior

The function

$$f(x) = \frac{x^2 - 9}{x - 3}$$

behaves as if it will be equal to 6 when $x = 3$. Of course, x cannot be 3, but its behavior as x approaches 3 is that $f(x)$ should be 6. Hence,

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6.$$

Limits As Behavior

This “behavior” explains why

- limits may not exist;
- horizontal asymptotes are limits as $x \to \pm\infty$;
- we can talk of the behavior of the average rate of change as the instantenous rate of change (the derivative)

Stop Spending the First Month on Review

- Summer packets of review material can work, but
  - if the kids already know the stuff, then it's just busy work;
  - if the kids don't know the stuff, it just frustrates them and increases their hatred of math; and
  - the kids still don't see *why* they need to know the stuff!

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  - the kids still don't see *why* they need to know the stuff!
- Review as you go when you need to
- Make review part of each homework assignment (and each test)
- Reviewing in August sends the signal that this class is like every other math class they've taken

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Use Differential Equations To Be Different

The first six days of my class

- **Day 1** Rates
- **Day 2** Exponential population growth
- **Day 3** Logistic population growth
- **Day 4** Position-velocity problem
- **Day 5** Euler's method
- **Day 6** Slope fields

Use Differential Equations To Be Different

Advantages of this approach

- Wildly different than any math class they've ever taken
- Opportunity to teach them calculator usage
- Reinforces that the method is as important as the answer
- Students who struggled in previous math classes tend to excel
- Introduces the need for more exact methods, and leads into differentiation
- Meaning of the derivative is crystal clear
- Applications drive the concepts

The Point of the AP Curriculum is Approximations

This idea is more obvious with BC than with AB:

- slope fields
- Euler's method
- graph and table problems
- Taylor polynomials
- Riemann sums
- numerous definite integrals only computable on a calculator
- emphasis on setting up integrals
- numerous functions which "model" a situation

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A cubic approximator for a function:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}f'''(x)(x-a)^3$$

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Only Simplify When It Helps

I consider these expressions simplified:

$$\frac{1}{1+(x-3)^2} \qquad \frac{7}{\sqrt{12}} \qquad \frac{x^3+4x^2}{x+1}$$

The operations one could perform does not help simplify the expression. However, writing

$$\frac{10+x^2}{5} \qquad \text{as} \qquad 2+\frac{1}{5}x^2$$

helps a great deal!

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This is also a correct statement:

$$\lim_{x \to a} \frac{x^2 - 9}{x - 3} = \lim_{x \to a} (x + 3), \quad \text{for all } a$$

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Only Simplify When It Helps

Simplification does not necessarily lead to a correct answer! The domain of

$$f(x) = \sqrt{x^4 - 16x^2}$$

is $(-\infty, -4] \cup \{0\} \cup [4, \infty)$, but simplifying leads to the function

$$f(x) = |x|\sqrt{x^2 - 16},$$

whose domain excludes zero.

Get to Derivatives of Trig Functions ASAP

Differentiating the functions

$$x^2 \sin x \qquad \frac{\cos x}{x^2} \qquad \sin\left(x^2\right)$$

justifies developing product, quotient, and chain rules.
Better justification (and less algebra) than

$$x^2(x+3)^5 \qquad \frac{x^2+1}{x^4} \qquad (x^2-x-1)^{10}$$

Get to Derivatives of Trig Functions ASAP

Order of presentation:

- Power Rule; Polynomials
- Sine and Cosine
- Product Rule
- New rule: $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{[f(x)]^2}$; use this to find Secant and Cosecant
- Quotient Rule; use this to find Tangnet and Cotangent
- Chain Rule

Use "exp" for Exponentials

We have all seen the following from our students:

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How can we get our students to stop making this silly mistake? By never using this notation until later in the course. Instead use the notation $\exp(x)$ for the exponential function.

Use “exp” for Exponentials

Students recognize that differentiating $\cos\left(x^2\right)$, $\ln\left(x^2\right)$ and $\sin\left(x^2\right)$ requires the chain rule.

To students, e^{x^2} looks like a power. With a change in notation, that mistake is corrected:

$$\frac{d}{dx}\left(\exp\left(x^2\right)\right) = 2x\exp\left(x^2\right).$$

Use “exp” for Exponentials

Exponentials with other bases:

$$\exp(x \ln 2) = e^{x \ln 2} = \left(e^{\ln 2}\right)^x = 2^x,$$

so other bases are just transformations of $\exp(x)$.

Hold Off Logarithms and Exponentials ALAP

Wait until after the FTC before dealing with logarithms. Then the function $\ln(x)$ may be developed so students understand it! From that, develop the exponential function.

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- is positive for $x > 1$ and negative for $0 < x < 1$;
- is unbounded so it's range is all real numbers.

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Consider $L(kx)$ for constant k. Then

$$\frac{d}{dx}[L(kx)] = \frac{d}{dx}\int_1^{kx} \frac{1}{t}\,dt = \frac{1}{kx}\cdot k = \frac{1}{x}$$

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so that $L(kx)$ is also an antiderivative of $\frac{1}{x}$. Hence, since two antiderivatives can at most differ by a constant, we know that $L(kx) = L(x) + C$. However, when $x = 1$, this becomes $L(k) = L(1) + C$. But we know $L(1) = 0$, so we have $L(k) = C$. Therefore,

$$L(kx) = L(x) + L(k).$$

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may also be derived from this definition.
Finally, the function L defined by

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

is called the *logarithm* of x. Note that there must be some value of x such that $L(x) = 1$ and we call this value e.

The Natural Logarithm

What about other bases?

- We want $f(x) = cL(x) = 1$ for a particular value of x. Call this particular x-value b. Then $cL(b) = 1$, or $c = 1/L(b)$.
- The number b is called the *base* of the logarithm. Hence,

$$f(x) = c\log_b(x) = \frac{L(x)}{L(b)}.$$

- Since b and c are related by $cL(b) = 1$, then when $b = e$, we have $c = 1$. Hence, $L(x) = \log_e(x) = \ln(x)$.

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The Exponential Function

- Let $\exp(x)$ be the inverse of $\ln(x)$. Then, by definition, if $\ln(a) = b$, then $a = \exp(b)$.
- As an inverse, $\exp(x)$ satisfies

$$\exp(\ln(x)) = x \qquad \text{and} \qquad \ln(\exp(x)) = x.$$

- Since $\ln(1) = 0$ and $\ln(e) = 1$, then when $x = 0$ and when $x = e$ we get

$$\exp(0) = 1 \qquad \text{and} \qquad \exp(1) = e.$$

The Exponential Function

- For reals a, b, and p and positive reals m and n, we let

$$m = \exp(a), \qquad n = \exp(b), \qquad \text{and} \qquad p = \ln(mn).$$

- Then

$$\ln(m) = a, \qquad \ln(n) = b, \qquad \text{and} \qquad \exp(p) = mn.$$

- Hence, $p = \ln(mn) = \ln(m) + \ln(n) = a + b$, or $p = a + b$.
- So then $\exp(p) = \exp(a + b)$.

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- Hence, $p = \ln(mn) = \ln(m) + \ln(n) = a + b$, or $p = a + b$.
- So then $\exp(p) = \exp(a + b)$.
- Also, $\exp(p) = mn = \exp(a)\exp(b)$.
- We have two expressions for $\exp(p)$. Equate them:

$$\exp(a + b) = \exp(a)\exp(b).$$

The Exponential Function

What is the derivative of $\exp(x)$? Begin by composing $\ln(x)$ with $\exp(x)$ in two ways.

- First, $\ln(\exp(x)) = x$.
- Second, we also have that $\ln(\exp(x)) = \int_1^{\exp(x)} \frac{1}{t}\,dt$.
- Therefore, $\int_1^{\exp(x)} \frac{1}{t}\,dt = x$.
- Taking derivatives of both sides, we get

$$\frac{1}{\exp(x)} \cdot \exp'(x) = 1, \qquad \text{or} \qquad \exp'(x) = \exp(x).$$

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The Exponential Function

Repeatedly using the property $\exp(a+a) = \exp(a)\exp(a)$ gives us the property $\exp(na) = (\exp(a))^n$.

- Let $a = 1$ in $\exp(a)^n = \exp(na)$.
- Recall $\exp(1) = e$.
- Then $\exp(n) = \exp(1)^n = e^n$.

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- Then $\exp(n) = \exp(1)^n = e^n$.
- This gives us another way to denote the exponential function $f(x) = \exp(x)$: $f(x) = e^x$.

Use Hyperbolic Functions

Advantages:

- Reinforces trig derivatives and antiderivatives
- Interesting applications
- Expected to know them for Calc II and Calc III
- In exponential form, they have appeared in the AP Exam

Ignore College Board

How did I come to these conclusions?
I stopped listening to College Board.

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Travesty! Recount such heresy before I call the AP Police!

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- Many calculus books: single- and multi-variable, real analysis
- MAA resources
- The history of calculus
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Then create a course *that makes sense* using the AP Course Description as a guide for minimal content

Erase the Word “Unit” from Our Vocabulary

- Planning by “units” is a convenience of administrators
- Speaking of “units” reinforces the student belief that math is disconnected, discrete chunks of unrelated material
- Do college mathematics professors think in “units”?

Write Better Tests

- Every test is a chance to do AP Exam preparation
  - Use multiple-choice and free-response
  - Part with a calculator and part without
  - Use old AP free-response on every test
- Do not use test banks!
- Do not let anyone else write your calculus tests!
- Do not allow kids to use their crutches. . .

Write Better Tests

Write the equation of the line tangent to the graph of $y = x^2 - 5x + 7$ at the point where $x = 3$.

A) $y = x + 2$
B) $y = -x + 2$
C) $y = 2x + 1$
D) $y = x - 2$
E) $y = -x - 2$

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The equation of the line tangent to the graph of $y = x^2 - 5x + 7$ at the point where $x = 3$ can be written in the form $y = mx + b$. Compute $m + b$.

A) -1
B) 0
C) 1
D) 2
E) 3

Consult Old Books

Find all points on the curve $y^4 - 5y^2 = x^4 - 4x^2$ at which the tangent lines are (a) vertical; (b) horizontal.

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Compute

$$\frac{dy}{dx} = \frac{2x^3 - 4x}{2y^3 - 5y}.$$

Then by the standard procedure there are vertical tangents at $(\pm 2, 0)$ and $(0, 0)$; and horizontal tangents at $(0, \pm\sqrt{5/2})$ and $(0, 0)$.

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Consult Old Books

A *singular point* is a point on a curve at which the derivative becomes 0/0. This indicates the presence of a *double point*—a point through which the curve passes twice. The tangent lines of the curve at a double point are found by considering only those terms of degree two in the equation and solving.

(Clyde Love and Earl Rainville, *Differential and Integral Calculus*, 5th edition, MacMillan, New York, 1954.)

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Since the terms of degree two are $-5y^2$ and $4x^2$, we ignore the other terms and solve $-5y^2 + 4x^2 = 0$ to get $y = \pm 2x/\sqrt{5}$. So the origin has no vertical or horizontal tangents.

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7. Why do we teach students to rationalize denominators?
8. Why do we still teach slope-intercept form?

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