Brackets or Parentheses?

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So here you are with your calculus class, teaching students how to excel on those wonderful problems that ask for extrema, critical points, and intervals of increasing and decreasing. Your lesson goes extremely well, and everyone has got it—congratulations! But then the students come into class the next day with their inevitable questions: "How do I know when to use brackets or parentheses on the intervals?" Another student says, "Yeah, on the very first problem, the one with $y = x^3 - 12x$, the interval where this is decreasing is from $x = -2$ to $x = 2$, but is the answer $(-2, 2)$ or is it $[-2, 2]$?" And another students pipes up with the common refrain "What is accepted on the AP Exam?" And that sparks some debate and more questions, and frankly, you're not sure yourself—about any of that!

Does that sound familiar to you? If so, that's pure speculation on my part that it does, because what I described was my class, about 10 years ago! The short answer is that the AP Exam Readers are not picky about brackets or parentheses for functions defined over all real numbers—there could be some execeptions based on domain issues, but in general, either are acceptable. However it brought to my mind the question concerning the actual use of brackets or parentheses: which is mathematically correct? I've had discussions about this with other calculus instructors which interested me enough to seek out some answers to this question, which I'd like to share with you.

A Definition

The most interesting thing I learned in my initial discussions and investigations was that the definition of an increasing or decreasing function has nothing to do with calculus. Yes, that's right, we need no calculus to define an increasing function. The definition of an increasing function is below.

> A function $f(x)$ *increasing* on an interval I if and only if $f(b) \geq f(a)$ for all $b > a$, where $a, b \in I$. If $f(b) > f(a)$ for all $b > a$, then the

function is *strictly increasing*.[1]

(There is a similar definition for a decreasing function, but we simplify our discussion by considering only increasing.) Three things about this definition jump out: equality is allowed; the lack of the condition of continuity; and there are no brackets or parentheses!

The possibility of equality implies that constant functions—their graphs are horizontal lines—are increasing functions. Of course, in our classes, we do not want to consider a function such as $f(x) = 9$ as increasing. So I arrived at the first thing I did incorrectly from a mathematical point of view: the notion of increasing functions that I taught my students was really the notion of *strictly* increasing. I had never said the phrase "strictly increasing" to my students. I simply said that a function like $f(x) = 9$ was neither increasing nor decreasing, which is true, depending on the definition you use. And that brought me to the fact that definitions matter! (This was a ridiculously slow realization on my part, since I am familiar with the deductive systems of geometries and abstract algebra, which always begin with axioms and definitions.) We should clearly define for our students the definition of increasing that we will be using in class. I personally chose to remove the possibility of equality from the definition of increasing that I use, for no other reason than to avoid constantly saying "strictly increasing" and we can just say "increasing."

The lack of continuity in the definition is an interesting aspect. Consider the function

$$g(x) = \begin{cases} x+1 & x < 0 \\ x+2 & x \geq 0. \end{cases}$$

This function is has a jump discontinuity at $x = 0$ and is therefore not differentiable at $x = 0$. However, by the definition, it is increasing on any interval.

As for the brackets versus parentheses question, consider this. The definition says that the property must apply for all $b > a$ where $a, b \in I$. Then consider the simple function $f(x) = x^3$ on the closed interval $[-1, 1]$. Take any point $b \in I$ such that $b > 0$. Then it is still true that $f(b) > f(0)$. Likewise, take any point $a \in I$ such that $a < 0$. Then it is also true that $f(0) > f(a)$. Then by the definition, this function is increasing on this interval, even though the derivative is zero at $x = 0$. But is it increasing on the closed interval or on the open interval? To see which, take any point $a \in I$ such that $a < 1$. Then we do indeed have $f(1) > f(a)$. Take any point $b \in I$ such that $b > -1$. Then we also have $f(b) > f(-1)$. Therefore, the function $f(x) = x^3$ is increasing on the *closed* interval $[-1, 1]$.

To be more precise in showing that $f(x) = x^3$ is increasing on this interval (indeed, on any interval), we should work with the inequality from the definition directly. Let $x = x_0$ and consider a point $x_0 + h$ where $h > 0$. Then we need

[1] From (with a change in notation) page 178 of G.H. Hardy's *A Course of Pure Mathematics*, tenth edition, Cambridge, 1952. Hardy uses the term "steadily increasing" instead of "strictly increasing." This definition also appears on page 6 (!) of J. Rogawski, *Calculus: Early Transcendentals*, first edition, 2008 and page 19 of J. Stewart, *Calculus*, seventh edition, 2012.

to show that $f(x_0+h) > f(x_0)$ for all x_0 and h. Note that since $h > 0$, then $x_0 + h > x_0$. Hence,

$$\begin{aligned} x_0 + h &> x_0 \\ (x_0+h)^3 &> x_0^3 \\ x_0^3 + 3x_0^2h + 3x_0h^2 + h^3 &> x_0^3 \\ 3x_0^2h + 3x_0h^2 + h^3 &> 0 \\ h(3x_0^2 + 3x_0h + h^2) &> 0 \\ h\left(3\left(x_0 + \frac{1}{2}h\right)^2 + \frac{1}{4}h^2\right) &> 0. \end{aligned}$$

This last inequality was obtained by completing the square, and the expression on the left-hand side is always positive for any x_0. Thus, $f(x) = x^3$ is increasing over any interval. Now this takes far too much time to do with the interesting functions we show students in calculus class, and I do not show them any of this. What we have instead is a short-cut using the derivative, and that is what they learn. However, this short-cut has a caveat, as many short-cuts do.

Some Calculus.

When we use the first derivative test to determine intervals of increasing or decreasing, we use the following result. This result can be proved using the definition of increasing and the Mean Value Theorem, with the assumption that f is differentiable.

> Let $f(x)$ be a continuous function. If $f'(x) > 0$ on an open interval (a,b), then $f(x)$ is increasing on (a,b).

By this result,[2] we see that the derivative of $f(x) = x^3$ is positive on the intervals $(-1,0)$ and $(0,1)$, and is therefore increasing on these intervals. Doesn't this contradict the definition of an increasing function? No, it does not. The result above *does not imply* that if a function is increasing on an open interval, then the derivative is positive on that interval. In other words, the result above is not an "if-and-only-if" statement. It is possible for a function to be increasing on an interval and to also not be differentiable, like the piece-wise function g above. It is also possible for a function to be increasing on an interval while the derivative is zero at a finite number of points on that interval, like the function $f(x) = x^3$. Moreover, we can say that $f(x) = x^3$ is increasing on the closed interval $[-1,1]$ because the closed interval meets the definition of an increasing function even though the result above applies only to open intervals.

[2] The proof is by contradiction. Assume $f'(x) > 0$ and that f is not increasing on I. Then there are $a, b \in I$ such that $f(a) > f(b)$. By the Mean Value Theorem, $f'(c) = (f(b) - f(a))/(b-a)$ for some $c \in (a,b)$. Since $f(a) > f(b)$, then $f(b) - f(a) \leq 0$. Since $a < b$, then $b - a > 0$. Hence, $(f(b) - f(a))/(b-a) = f'(c) \leq 0$. But this contradicts the assumption that $f'(x) > 0$ on I. The result follows.

An Example.

Let's find the intervals where $h(x) = (x-1)^3(x-5)$ is increasing and intervals where $h(x)$ is decreasing. Following the first derivative test, we take the derivative and set it to zero:

$$\begin{aligned} h'(x) = 3(x-1)^2(x-5) + (x-1)^3 &= 0 \\ (x-1)^2(3(x-5) + x - 1) &= 0 \\ (x-1)^2(4x-16) &= 0 \\ 4(x-1)^2(x-4) &= 0 \\ x &= 1, 4. \end{aligned}$$

Since $h'(x) < 0$ for $x < 1$ and $1 < x < 4$, and $h'(x) > 0$ for $x > 4$, h is decreasing on $(-\infty, 1)$ and $(1, 4)$, and increasing on $(4, \infty)$. However, where does this leave $x = 1$ and $x = 4$? The case where the derivative is 0 is not covered by the first derivative test: the test says that *if* the derivative is positive (or negative), *then* we may conclude something about the function increasing (or decreasing). When the derivative is neither positive nor negative, *then we cannot use the first derivative test*, and we must appeal to the definition of increasing (or decreasing). This reasoning implies that h is decreasing on $(-\infty, 4]$ and increasing on $[4, \infty)$. Note that $x = 1$ can be legitimately placed within the interval $(-\infty, 4]$, and we can include $x = 4$ in both intervals.

This may seem like a horrible mistake to include $x = 4$ in both intervals! How could $x = 4$ be both increasing and decreasing? Well, it cannot, but not for the reason you may think. The definition of increasing (and decreasing) applies to *intervals*, not *points*. So it is incorrect to refer to a function increasing at a single point. Hence, this function does not increase or decrease *at the single point* $x = 4$, and it also does not increase or decrease at any other single point, either.[3]

With that aside, consider the definition. We do have that for any $b \in [4, \infty)$ where $b > 4$, we also have $h(b) > h(4) = -27$; so h is increasing on this interval. Likewise, for any $a \in (-\infty, 4]$ where $a < 4$, we also have $h(a) > h(4) = -27$; so h is decreasing on this interval. Thus, $x = 4$ can be correctly placed within both intervals and we can use brackets on that side of the interval.

That is so much to think about for the typical calculus student, and that is a good reason why the AP Exam Readers would accept either brackets or parentheses.

[3] Yes, there are many questions in many books where questions are posed like "Is f increasing at $x = 5$?" In this case, we should just narrow our eyes, glance away, and understand that the question is really asking "Is f increasing on the interval $[5 - \epsilon, 5 + \epsilon]$ for some small $\epsilon > 0$?" Indeed, this is equivalent to E. Landau's definition of "increasing at a point" where the point in question is the center of an interval of width 2ϵ. This appears on page 88 in Landau's *Differential and Integral Calculus*, third edition, AMS-Chelsea Publishing, 1960.

Concavity.

If we know that the function in question has a second derivative, then a necessary and sufficient condition for concavity is determined by the second derivative. This gives us the "rule" we teach our calculus students:

> If $f''(x)$ exists and $f''(x) \geq 0$ in $[a,b]$, then $f(x)$ is concave up on $[a,b]$. If $f''(x)$ exists and $f''(x) \leq 0$ in $[a,b]$, then $f(x)$ is concave down on $[a,b]$.

Again, notice the explicit use of the closed interval. However, you may remember that with increasing and decreasing, we really use these terms to mean *strictly* increasing and *strictly* decreasing. We have the same situation here. What we traditionally discuss is that a function is concave up when $f''(x) > 0$, which means we are actually talking about a function being *strictly* concave up. We could make the decision to always mean "strictly concave up" when we say "concave up." Therefore, using this convention, we would not accept a function being concave up on $[7,\infty)$, but only on the interval $(7,\infty)$. So it seems when it comes to concavity, we can decide firmly the question of brackets or parentheses: we always use parentheses. Oh, but there's a twist: consider another definition of concavity.

> If $f'(x)$ exists and is increasing on an interval I, then f is concave up on I. If $f'(x)$ exists and is decreasing on an interval I, then f is concave down on I.[4]

This definition has some interesting implications as well: f must be differentiable to have concavity and the points of inflection are the endpoints of the increasing or decreasing intervals of f'. The second derivative only tells us concavity on the open interval, but this definition, which is related to the definitions of increasing and decreasing, implies concavity can occur on the closed interval. We will find the concavity of $h(x) = (x-1)^3(x-5)$.

$$\begin{aligned} h''(x) = 8(x-1)(x-4) + 4(x-1)^2 &= 0 \\ 4(x-1)(2(x-4) + x - 1) &= 0 \\ 4(x-1)(3x-9) &= 0 \\ 12(x-1)(x-3) &= 0 \\ x &= 1, 3. \end{aligned}$$

We find that $h''(x) > 0$ for $x < 1$, $h''(x) < 0$ for $1 < x < 3$, and $h''(x) > 0$ for $x > 3$. This implies that $h'(x)$ is increasing for $(-\infty, 1]$ and $[3,\infty)$ and therefore h is concave up on these intervals. We also have that $h'(x)$ is decreasing on $[1,3]$, and therefore h is concave down there.

[4] From page 207 of Finney, Demana, Waits, Kennedy, *Calculus: Graphical, Numerical, Algebraic*, third edition, Pearson, 2007. This definition also appears on page 239 of J. Rogawski, *Calculus: Early Transcendentals*, first edition, 2008.

Once more, we have a seemingly incorrect statement that h is both concave up and concave down at $x = 1$ and $x = 3$. However, concavity, like increasing/decreasing, refers to intervals, not points. If we are to be consistent with our use of closed intervals for increasing/decreasing, then we should be consistent with closed intervals for concavity, if we use the definition of concavity based on an increasing/decrasing derivative. Definitions matter! As another example of this, consider another definition of concavity.

> If the graph of $f(x)$ lies above all of its tangents on an interval I, then it is concave up on I. If the graph of $f(x)$ lies below all of its tangents on I, it is concave downward on I.[5]

Are these intervals open or closed under this definition? Consider an endpoint of one of these intervals, and assume the function f has an inflection point at this endpoint. Then there is a tangent line at this endpoint, but this tangent cannot lie entirely above or below the function. Hence we would not include the endpoint in an interval describing concavity, and thus the interval is open. Under this definition, we would need to use parentheses every time we describe an interval of concavity.

Concavity without Calculus.

But all of the preceeding statements of concavity rely on calculus. There is a definition of concavity which does not rely on calculus, just as there is one for increasing. That definition is below.

> A continuous function $f(x)$ is concave up on an interval I if, and only if, for any $a, b \in I$, and any t such that $0 < t < 1$, $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$. Likewise, f is concave down if, and only if, $f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$.[6]

We can show, using the definition, that $f(x) = x^2$ is concave up on any interval. Let $a, b \in \mathbb{R}$ and let $t \in (0, 1)$. Then

$$
\begin{aligned}
f(ta + (1-t)b) &\geq tf(a) + (1-t)f(b) \\
(ta + (1-t)b)^2 &\geq ta^2 + (1-t)b^2 \\
t^2a^2 + 2t(1-t)ab + (1-t)^2b^2 &\geq ta^2 + (1-t)b^2 \\
(t^2 - t)a^2 + 2t(1-t)ab + ((1-t)^2 - (1-t))b^2 &\geq 0 \\
-t(1-t)a^2 + 2t(1-t)ab - t(1-t)b^2 &\geq 0
\end{aligned}
$$

[5] From page 216 of J. Stewart, *Calculus*, seventh edition, 2012.

[6] From (with a change in notation) page 61 of W. Rudin, *Real and Complex Analysis*, third edition, 1987. Rudin uses the term “convex” to describe functions which are concave up.

Since $0 < t < 1$, we can divide both sides by $-t$ and $1 - t$.

$$\begin{aligned} a^2 - 2ab + b^2 &\geq 0 \\ (a-b)^2 &\geq 0. \end{aligned}$$

The last inequality is always true for all a and b, and since each step is reversible, we have the established $f(ta + (1-t)b) \geq tf(a) + (1-t)f(b)$. This proves $f(x) = x^2$ is concave up everywhere. Once more, however, this is too much for the typical caluclus student. Better may be a translation into more common words. In English, that definition says the following.

> A continuous function $f(x)$ is concave up on an interval I if, and only if, for any $a, b \in I$, the values of the function on $[a, b]$ do not exceed the values of the line segment connecting $f(a)$ and $f(b)$. If the values of the line segment connecting $f(a)$ and $f(b)$ do not exceed the values of the function on $[a, b]$, the function is concave down.

This implies a picture: a secant line between two points on $f(x) = x^2$ will always be above the graph of the parabola. However, notice that the explicit use of the closed interval and the phrase "does not exceed" implies that points of inflection may be included in intervals of concavity.[7] Thus, if a function has a point of inflection at $x = 7$ and is determined to be concave up for $x > 7$, then we may correctly say that the function is concave up on $[7, \infty)$. Again, you can see why the AP Exam Readers are given leniency concerning brackets or parentheses!

Domains.

The most vital consideration between brackets or parentheses is the issue of domains. Simply put, if the function is continuous at the endpoint of an interval of increasing/decreasing, or concavity, I urge my students to use a bracket. Otherwise, it gets a parenthesis. For example, consider the function $j(x) = x\ln(x)$. The derivative given by the product rule is $j'(x) = \ln(x) + 1$. Setting $j'(x) = 0$ yields $x = 1/e$. Thus, $j'(x) < 0$ for $0 < x < 1/e$ and $j'(x) > 0$ for $x > 1/e$. Since $j(x)$ is not defined for $x \leq 0$, the interval must be open on the left of our decreasing interval: $(0, 1/e]$. We also have $j(x)$ is increasing on $[1/e, \infty)$.

Conclusion.

Should we use brackets or parentheses? The AP Exam generally accepts either—as long as the stated interval does not contradict the domain of the function—so students shouldn't worry about it on the AP Exam. But we should be teaching

[7] If $(a, f(a))$ is a point of inflection, then the segment from $f(a)$ to $f(b)$ is larger than f everywhere between a and b, except at $x = a$ where the values are equal. Hence, the values of the line do not exceed the function.

good mathematics, not just test preparation. The reliance on the first derivative test gives a students a false impression that “increasing” and “the derivative is positive” are identical properties when they are not. This leads to the confusion concerning brackets or parentheses. Moreover, this could also lead to misconceptions in later calculus courses, or could lead to incorrect answers with stranger functions (like the piecewise example above). Second, you must decide for yourself and your students whether to insist on brackets or parentheses. I personally insist my students use brackets whenever possible (but I accept either on assessments). This forces them to think about the domain of the function and reduces the likelihood that they will confuse an open interval with a point. Also, in mathematics, we generally strive to use the most inclusive definition possible. Including the endpoints when possible is the more inclusive approach.