# The Evolution of GCTM's State Math Tournament

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## 1 Part One: The Tournament's Establishment and Structure

### 1.1 A Little History

Math Tournaments for high school students existed in the state of Georgia since the 1950s. The first tournament for which we have documented evidence is a math competition at Georgia Tech in 1958. The first-place prize winner received a color television set. We also have documented evidence of another tournament at Georgia Tech the following year (the winner receiving a medal and cash instead of a television), but no other documentation of any state-wide or regional math tournament in Georgia until the 1960s. In the mid-1960s Lanier High School in Bibb County hosted a tournament open to all high schools in middle Georgia. There was also a Fulton County high school math tournament, held at Briarwood High School, in March 1972. We do not know if either of these tournaments in Bibb or Fulton counties were one-time events or, if they were the first of annual tournaments.[1]

There is no other evidence of a state-wide math tournament until the inaugural Georgia Southwestern College Math Tournament in 1974.[2] As the story goes, Dr. William Kipp and Dr. Jay Cliett were at a meeting of the Southeastern Section of the MAA at Samford University in March 1972, where they heard of the math tournaments then occurring in Alabama. They both decided at that meeting that Georgia should have a big state-wide math tournament for high schools. They were unaware of the others that had taken place in Georgia prior to the 1970s,

[1] Most of the historical information in this paper is from two sources. The first is the fascinating document "The Early History (to 1984) of the Georgia Council of Teachers of Mathematics", pp. 58-63, posted at `https://www.gctm.org/whoarewe` (retrieved July 25, 2020). The second source is the files of GCTM's VP for Competitions

[2] Although Augusta College inaugurated a math tournament in 1974 as well, participation was limited to Richmond County schools. Georgia State University held a math tournament on their main Atlanta campus in the mid-1970s as well. By the way, we are using the names of the colleges and universities as they were at the time.

and so were under the impression that they had created the first state-wide tournament in Georgia. It is still an annual event, and is the longest running state-wide tournament.[3]

However, GCTM has evidence of regional math tournaments taking place prior to 1974. It is believed that other institutions held math tournaments open only to schools in their county or local area. These include West Georgia College's Math Day for schools in Carroll County, and local tournaments organized and run by high school teachers and administrators for their system schools. The justification for this assertion lies in the documents discussing the implementation of a state math tournament held by GCTM, indicating that there are already regional math tournaments.

The most important of the aforementioned documentation is a letter from GCTM's first president, Gladys M. Thomason, addressed to the Executive Committee of GCTM, from September 1972, in which she urged GCTM to start a state math tournament championship. Three years of discussions among GCTM leadership followed. In 1975 a "special projects" committee was tasked with figuring out how to implement a state math tournament.[4] Finally, the first state math tournament happened on April 30, 1977 at Northeast High School in Macon.

GCTM has always maintained oversight of the State Math Tournament on behalf of the Georgia High School Association. Indeed, we have permission from GHSA (granted in a letter from GHSA dated March 23, 1978) to allow non-GHSA schools to attend and compete, as long as we offer separate awards to GHSA schools and non-GHSA schools. This allows GCTM to promote mathematics competitions to every school—and therefore to every student—in the state. We currently give awards to the top 5 schools overall, regardless of GHSA or non-GHSA classification. This does not violate the GHSA parameters, because we also award "classification champion" to the best school falling outside the top 5 in each of the (currently, seven) GHSA classes, as well as a separate non-GHSA class. By awarding the best in the non-GHSA class, we are able to allow teams from GISA schools, non-GISA private schools, homeschools, and one year, a middle school, to compete.[5]

The 1978 and 1979 State Math Tournaments (which we shall abbreviate SMT for the rest of this paper) took place at Northeast High School in Macon, and then moved to Mercer University's Macon campus in 1980. There, the SMT stayed until 2007. In 2008 and 2009, the SMT was held at Wesleyan College in Macon. While Wesleyan was very gracious, the space was too small. In 2010, the SMT found a larger home at Macon State College, where it has remained since (although now the institution has become Middle Georgia State University).

In 1982, GCTM fixed the date of the SMT to take place on the last Saturday in

[3]This information is the author's recollection of the history of the tournament from Dr. Kipp's reminiscences at the 2017 GSW tournament, the first after Dr. Cliett's passing.

[4]This is detailed in "The Early History" mentioned in the first footnote. As an aside, this special projects committee was chaired by Dr. Hiram Johnston, who, 17 years later, was my math education professor at Georgia State!

[5]Yes, this middle school actually did participate in the high school Varsity level at other tournaments around the state, placing very well, earning their invitation to the state math tournament

April each year.

## 1.2 Selection of Teams

Schools are invited to send four students to the SMT representing their school. The selection of the four students is up to the school; the math team sponsor or other school-level personnel makes the decision. Schools must send a faculty representative to accompany the student team. Schools are invited based on a point-system. Points are earned after each qualifying tournament, calculated by dividing the number of schools present at the tournament by the schools overall rank. For instance, in a tournament of 20 schools, a school placing 4th gets $20/4 = 5$ points. The highest single point values earned by schools at any qualifying tournament they attend becomes their "qualifying index". The schools are then ranked by qualifying index and the top 40 schools get invited to the SMT. Of course, the tournaments, the ranks, and the index are all based on high school Varsity-level tournaments; junior varsity or middle school tournaments do not factor into this index.[6] The Tournament Secretary receives all results from tournament organizers and calculates the points by April 1 each year. There is no "point cutoff". The cutoff occurs when the limit of 40 schools is reached.[7]

The selection of schools to be invited was not always like this. In 1977, the selection was made through qualifying exams given in ten locations around Georgia. The top two schools in each of the 10 regions went on to the SMT, along with the top 25 teams from other tournaments taking place around Georgia.[8] By 1979, there were so many tournaments in Georgia, the qualifying exams were removed, and the top four teams from the eight largest regional tournaments were invited to the SMT. Then, in 1981, the rules were changed again due to the proliferation of tournaments: the top two schools at each tournament where at least five schools participated were invited to the SMT.[9] This rule stayed in place until the late 1990s, when GCTM started using the qualifying index.[10] In 1982, GCTM determined that any qualifying tournament should take place on or before March 31 each year.

The SMT itself is used as a selection process. Since 1989, Georgia has sent a team of students representing the state to a national (now international) math tournament called the American Regions Math League (ARML). Each team of students at ARML represents a contiguous geographic region of the U.S. which could be a state (like the Georgia team), a county (like the Montgomery County, Maryland team), a city (like the New York City team), or any other well-defined, agreed upon

[6] In-school contests such as the AMC, the Georgia Math League, and the Kennesaw State University Contest also do not factor into the qualifying index.

[7] Although a qualifying index greater than or equal to 4 generally results in an invitation.

[8] The top 25 was determined cumulatively by rank; i.e., all first place teams, then all second place teams, etc., until 25 distinct teams were invited.

[9] More details are in "The Early History".

[10] Jeff Floyd, teacher at Woodward Academy, had this "qualifying index" idea. He believed that teams who consistently placed 5th or 6th at tournaments where over 25 schools compete—which indicates they are solid, competitive teams—never got invitations to the SMT under the old procedure. David Hammett subsequently implemented the qualifying index scheme.

boundary.[11] Each of these regions may send as many teams of students as they like; each team consists of 15 students. Georgia currently sends three teams of 15 students each. These students are selected at the SMT. By the way, the Georgia team placed first in the ARML A division in 1992 and first in the ARML B division in 2019.

The selection of the Georgia ARML team led to an expansion of another rule concerning the SMT: individual qualification. In 1977, the top 2 individuals on the written qualifying test from each region were invited to the SMT, for individual (not team) award consideration. These individuals could have also been members of the invited teams, or not. As the years progressed, this rule became simply the top Varsity winner at every qualifying Varsity tournament got an individual invitation. The coaches of Georgia ARML team are interested in obtaining the best possible team, but they need a way to compare student performance among the best in the state. Thus, the SMT committees have allowed the Georgia ARML coaches to invite up to 20 individuals who show demonstrated ability, but are not a member of an invited team, and who have not placed first at a Varsity tournament. These additional invitations are at the discretion of the Georgia ARML coaches, and at the mercy of space and seating limitations imposed by the facility, GCTM, and the SMT committee.[12]

## 1.3 The Format

The format of the SMT was initially three rounds: a test, ciphering, and pair ciphering. The individual test consisted of 50 multiple-choice problems in a time-limit of 90 minutes. The individual ciphering consisted of 10 problems given one-at-a-time, in which students write their answers and submit them within 2 minutes per problem. The pair ciphering consisted of three rounds of 4 problems each. Students from the same school could work in pairs to submit answers to each round within four minutes. No calculators were allowed at all.

In 1994, the first change in format occurred with the introduction of open-ended questions on the written test. Instead of 50 multiple-choice problems, there were now 45 multiple-choice problems and 5 open-ended problems. Students had to write their answers to the last five problems. The test currently has this same format.

In 1995 came the next change: graphing calculators! Calculator use was allowed on the written test only. (The ciphering and pair ciphering remained calculator-prohibited.) This change was made to encourage teachers and students to use the soon-to-be-prevalent graphing calculators. Calculator usage also necessi-

[11] Team organizers must agree on the boundary. For instance, the Eastern Massachusetts team and the Western Massachusetts team have such an agreement, as do the Upstate New York team and the New York City team.

[12] For these "ARML consideration" invitees, the Georgia ARML coaches look at the student's placement in Varsity tournaments, scores on the AMC and the AIME, and they have been known to consider first place wins in JV tournaments, first place wins in middle school tournaments, and state MathCounts winners. The key factor in these invitations is demonstrated ability at the Varsity level, not "I think this student has potential."

tated some changes in the types of problems students were asked to solve on the exam. Graphing calculator usage continues to be allowed on the written.

The last change in format happened in 2012. In an effort for more team interaction, the pair ciphering was replaced with a team round. The team round consists of 12 problems given to the four-person team at once, and they have 20 minutes to submit their answers to the 12 problems. Calculators are not allowed on the team round.

Since 2010, all invited participants and their coaches receive a free SMT T-shirt, and—of course!—plenty of refreshments.

It is the responsibility of the VP for Competitions[13] to form a committee to write all the problems used at the SMT, and to help run the event on the day. The SMT Committee consists of six at-large members (serving a two-year term), the Tournament Secretary, and the VP for Competitions. The six at-large members of the committee can be anyone interested in writing problems and in helping run the tournament. Previous SMT Committee members have been high school teachers, middle school teachers, college and university professors, college students who were alumni of the SMT, and alumni of the SMT who are now in business and industry.[14] The entire event—the facility, the printing, the refreshments, the T-shirts, the trophies—is funded by GCTM, and the SMT Committee are all volunteers. If you are interested in volunteering, please send an email to `competitions@gctm.org`.

## 1.4 Impact

When our members think of GCTM, the majority probably think of the Georgia Math Conference at Rock Eagle first. That makes sense, because it is our largest, longest-running, and most successful annual event. However, some may think first of our highly-successful Summer Academies. Still others may think of our advocacy program "Math Day at the Capitol" or even the very journal you are reading right now. But I believe few think of the SMT. Of our visible initiatives, events, and activities, only the Georgia Math Conference has a longer history.[15] The SMT has played an important role in the stimulation and development of mathematics competitions in Georgia. I have spoken with math team sponsors in many other states, and only Alabama and Florida come as close having as many annual regional, local, and state-wide math tournaments, but still not as many as in Georgia. I believe this growth in math tournaments is a result of the decades-long encouragement of GCTM. This growth extends in a small way to GCTM itself: I know of a few teachers who have become members of GCTM as a result of sponsoring their math teams at the SMT.

But another impact of the SMT is on our students. The SMT is the culminating event of a school year filled with the struggle to conquer interesting non-curricular

[13] The author held this position for five two-year terms from 2009 to 2019.

[14] Most recently, Dr. Walter Sun contributed problems. He placed first at the 1990 and 1991 SMTs, and is currently working at Microsoft as the lead developer of the search engine Bing.

[15] Although, in fairness, *Reflections* was established in September 1972, when the idea of a SMT was first introduced. This information also comes from "The Early History", pp. 56-57.

mathematical concepts of number theory, combinatorics, and advanced Euclidean geometry, as well as trigonometry, algebra, logarithms, functions, and conic sections. The SMT is a day of excitement and challenge, and discussion among peers of interesting problems. The awards ceremony is highly anticipated and charged with excitement. Some students, however, view the awards as only a prelude to the announcement of who will be asked to join the Georgia ARML team—the most selective group of competitive math students in Georgia.

As a result, the SMT is the most visible program of GCTM to our students. (Indeed, some students call the SMT simply "GCTM" in much the same way we may call the Georgia Math Conference simply "Rock Eagle".) Since it is GCTM's mission to provide every student with high-quality mathematics education, the mission must also include enriching those mathematically-talented students who will later go into industry, business, teaching, and academia. Therefore, I would argue that we want our students to know something of our organization so they are encouraged to join, contribute, and volunteer to GCTM later in their careers. The SMT serves in this important and often overlooked capacity. Don't you want to be a part of this?

# 2 Part Two: The Tournament's Mathematics

In this second part of the article, we will look at the mathematical content of the SMT problems. The SMT problems were originally a reflection of the expectations of the knowledge of a good high school student, and so the SMT was a celebration of that knowledge. Over the years, the SMT has changed. In this second part, we will describe some of those changes.

## 2.1 Some Background

The written test currently consists of 45 multiple-choice questions and 5 free-response questions. Students receive five points for each correct answer, and they receive one point for each problem unanswered. Incorrect answers earn zero points. The awarding of points for blanks inhibits random guessing on multiple-choice questions. There are also ciphering problems. Students are given two minutes to submit a correct answer to one ciphering problem (there are 10 total ciphering problems). Before 2012, there were also a pair ciphering round. These were given to pairs of students in groups of four problems to answer in four minutes. Since 2012, this has been replaced with a team round. The team round is 12 problems given to the four-student team to complete in 20 minutes. At the end of 20 minutes they submit all 12 answers. In this article, however, we will use written test problems to illustrate the mathematical topcs.

The multiple-choice test is currently created under the following guidelines. It is suggested that the test have a certain range of numbers of problems per topic. Table 1 shows the details. Note that these are guidelines, not requirements. The categorization of topics is determined by the members of the SMT Committee.

| Topic | Number of Problems |
|---|---|
| Algebra | 10–18 |
| Geometry | 8–12 |
| Analytic Geometry | 5–10 |
| Trigonometry | 4–8 |
| Precalculus | 4–8 |
| Calculus | 1–3 |
| Discrete Math | 10–15 |

**Table 1:** Current written test topic breakdown

For some problems which may fall into two or more categories a single category is chosen.

The early years of the written test reflect a different emphasis in the types of problems. Compared to the current breakdown, there were fewer "discrete math" problems. Included in the current definition of "discrete math" problems is any problem concerning statistics. Currently, there tend to be as many statistics problems as calculus problems, and this is partly due to the prevelence of AP Statistics and partly due to the changes in the high school math curriculum since the mid-2000s. It is therefore interesting to discover that the early SMT tests contain fewer discrete math problems, and also virtually no statistics problems. The early tests, however, contain far more calculus, analytic geometry, and traditional trigonometry problems than the more recent tests.

One final note: the problems increase in difficulty from number 1 to number 45. Thus, the proposed difficulty in a test question is reflected in the problem number; the greater the problem number, the greater the difficulty. This "ranking" of difficulty is the determination of the SMT Committee. The Committee is not always accurate in their assessment, but they generally provide a test that has a smooth transition from easy to difficult. The difficulty level "re-sets" for the free-response test problems. Problem 46 is easy and then subsequent four problems increase in difficulty to problem 50. The result is that problems 46 through 50 may be easier than problem 45. Item analysis of the test and ciphering problems has been performed every year since 2012, and the results reported in that Summer or Fall issue of *Reflections*, should you want to know more on the accuracy of the Committee's difficulty rankings.

## 2.2 Consistent Topics

Some things never change. On every SMT test, there appear a variety of algebra and geometry problems. Many of the algebra and geometry problems from 40 years ago could very well be similar to problems appearing on the test just a few years ago. Table 2 lists the top five categories of problems included in SMT

| Topic | Percentage 1982-1990 | Topic | Percentage 2011-2019 |
|---|---|---|---|
| Geometry | 26.3% | Geometry | 19.3% |
| Trigonometry | 10.9% | Counting/Probability | 10.4% |
| Polynomials/Functions | 9.4% | Sequences/Series | 10.2% |
| Analytic Geometry/Conics | 9.3% | Trigonometry | 9.6% |
| Counting/Probability | 7.2% | Polynomials/Functions | 9.0% |

**Table 2:** Topics with the Largest Percentage of SMT Problems

problems over the years 1982-1990 and 2011-2019.[16] We see that over a quarter of the 540 problems from 1982-1990 concern geometry, and almost a fifth of the 644 problems from 2011-2019 are on geometry.[17] Trigonometry appears to be another perennial favorite of SMT test writers, as is analytic geometry, conic sections, counting, and probability. You may be surprised that "algebra" is not a category. That is because, in this writer's opinion, algebra is used so often that nearly every problem on the SMT requires some algebra skills. But that does not mean that the style of the algebra problems has not changed! The problem below is a typical problem in algebra that requires algebraic manipulation.[18]

**1987, #15.** The solution to $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = 3$ is

**(A)** $\frac{4}{5}$ **(B)** $\frac{5}{4}$ **(C)** $\frac{3}{5}$ **(D)** $\frac{5}{3}$ **(E)** 3

The student who is careful with arithmetic could plug in the answer choices and find the correct answer. However, a problem such as the next one defies that process.

**2007, #20.** When $(1-2x)^3(1+kx)^2$ is expanded, two values of $k$ give the coefficient of $x^2$ as 30. The sum of these two values of $k$ is

**(A)** $-1$ **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

Some algebraic manipulation is required, but multiplying the expanded cube and the expanded square together is not necessary. Note also that the answer choices cannot be "plugged in"! The style of the algebra problems has changed over the years, from rote symbolic manipulation to clever use of the understanding of how the manipulations work.

[16]It should be noted that the SMTs from 1977 through 1981 are missing. This is why we begin our investigation with 1982. If you may know where a copy of any of these tests may be hiding, please let someone at GCTM know!

[17]This categorization includes all Euclidean plane geometry and solid geometry, but not any problem which also involves trigonometry.

[18]Solutions to all the problems presented appear in the Appendix to this paper.

One algebraic problem that shows up so much that it is almost a tradition at the SMT is what we call the "defined operation" problem. That is, a brand new binary operation is defined and the students are asked to use it immediately.

**1989, #6.** For all nonnegative real numbers $a \neq b$, $a * b$ is defined by

$$a * b = \frac{a^2 - b}{a - b}.$$

If $5 * x = 11 * 0$, then $x = ?$

**(A)** 25 **(B)** 11 **(C)** 6 **(D)** 4 **(E)** 3

Twenty-one years later, we have another "defined operations" problem that is noticeably more difficult, even though it is placed at Number 13.

**2010, #13.** On positive real numbers, we define the operation $x \bowtie y$ to be

$$x \bowtie y = \frac{x^2 + y^2}{2xy}.$$

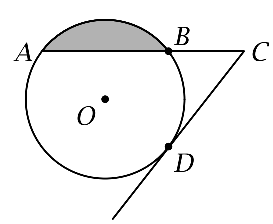
There are two values of $z$ such that $((1 \bowtie 2) \bowtie 3) \bowtie z = \frac{5}{4}$. Find the positive difference in the two values of $z$.

**(A)** $\frac{169}{240}$ **(B)** $\frac{169}{80}$ **(C)** $\frac{338}{45}$ **(D)** $\frac{169}{20}$ **(E)** $\frac{338}{15}$

Geometry problems on the SMT have changed very little over the years. Almost any of the geometry problems appearing on any SMT test would make excellent SMT problems now or in the future. This is what we mean by "consistent topics": geometry will always be part of high school mathematics and so will always be part of the SMT. Geometry problems are also the most resistant to the allowed use of graphing calculators.

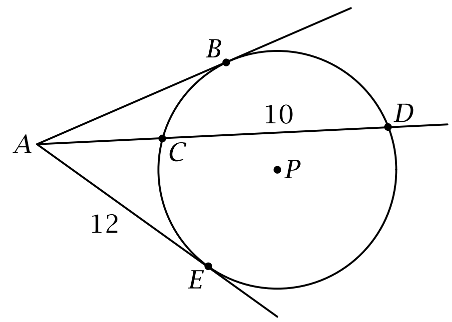
As examples of the abundant geometry problems, we offer four problems. The first two follow, and they involve tangents to circles. Notice that they are of comparable difficulty, but the older one is placed at Number 43 and the newer one is placed at Number 24.

**1982, #43.** Circle $O$ has radius 12, $BC = 4$, and $DC = 8$, and $\overline{CD}$ is tangent to circle $O$. Find the area of the shaded region.



**(A)** $24\pi - 72$ **(B)** $144\pi - 18\sqrt{3}$ **(C)** $\dfrac{14\pi - 18\sqrt{3}}{3}$
**(D)** $24\pi - 36\sqrt{3}$ **(E)** None of these

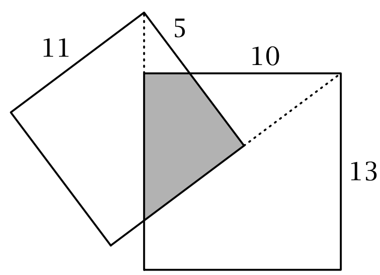
**2016, #24.** In the diagram below, $\overline{AE}$ and $\overline{AB}$ are tangent to circle $P$, and a line from $A$ intersects circle $P$ at $C$ and $D$. Given that $AE = 12$ and $CD = 10$, compute $AC \cdot AB$.




**(A)** 24 **(B)** 48 **(C)** 72 **(D)** 96 **(E)** Cannot be determined

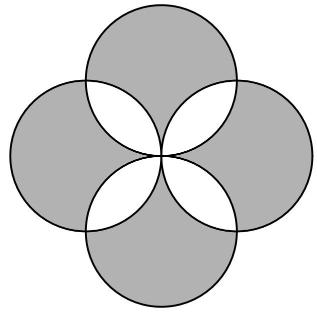
The next two are both area problems, 25 years apart.

**1984, #27.** Given the intersecting squares with lengths as indicated in the figure below, compute the area of the shaded lozenge.




**(A)** $\dfrac{315}{8}$ **(B)** 39 **(C)** $\dfrac{307}{8}$ **(D)** $\dfrac{507}{8}$ **(E)** $\dfrac{363}{8}$

**2019, #34.** Sarah draws a four-leaf clover by shading portions of four overlapping circles of radius 3.25 cm as shown below. Compute the total area, in square centimeters, of all the shaded regions.



**(A)** 82.5 **(B)** 84.5 **(C)** 86 **(D)** 86.5 **(E)** 98

Geometry problems stand the test of time. They are also of similar (no pun intended) style and format through the years. This is not true of other topics.

## 2.3 Disappearing Topics

The shift in emphasis of certain topics in high school mathematics is evident when examining over 40 years of SMT tests. Indeed, in examining Table 2, we find a list of typical high school mathematics topics from the 1980s—except perhaps the emphasis on counting and probability—and this is reflected in the content of the test. However, examining the tests over the nine-year period from 2011 to 2019, we find a different story. Certainly geometry, trigonometry, and polynomials are still included, but the analytic geometry/conic section category has been reduced. This may make little sense in that analytic geometry and conic sections are still part of our existing high school math curriculum. But the focus of the questions has shifted. Consider the following problem from 1983.

**1983, #30.** How many points do the graphs of the equations $x^2 + y^2 = 25$ and $y = x^2$ have in common?

**(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

This question is easier than its placement at Number 30 indicates. It is made even easier with a graphing calculator, which explains why we do not see problems like that today. Indeed, this is the kind of conic section problem we see in the 2010s.

**2011, #35.** The point $P\left(7, \frac{21}{4}\right)$ lies on the hyperbola defined by

$$\frac{(x-2)^2}{16} - \frac{(y-3)^2}{9} = 1.$$

What is the *sum* of the distances from $P$ to the foci of the hyperbola?

**(A)** $\frac{1}{4}\sqrt{593 - 160\sqrt{7}}$ **(B)** $\frac{1}{4}\sqrt{593 + 160\sqrt{7}}$

**(C)** $\frac{41}{4}$ **(D)** $\frac{25}{2}$ **(E)** $\frac{55}{4}$

There are other types of problems no longer appearing on the SMT that were prevalent 30 years ago. One type of problem not seen on the SMT tests in recent years is the following.

**1983, #26.** What is the value printed by the program below?

```
10 LET A = 1
20 LET B = 2
30 FOR N = 1 TO 5
40 LET A = B * A
50 LET B = A / B
60 NEXT N
70 PRINT B
80 END
```

**(A)** 2 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** 32

There was a computer programming or computer algorithm problem on nearly every SMT until 1991. They were so recognizable that there is not even a statement in the problem that this program is written in the BASIC computer language. Another programming problem appeared in 1996, and the last one appeared in 2001. These kinds of problems are now nonexistent, although with the increase in the number of students taking AP Computer Science, there could be a case made for bringing them back. The skills required to solve such problems—logic, recursion, algorithmic reasoning—can still be found on the SMT, just not in the context of pseudocode.

Another example of a disappearing topic is calculus problems. While calculus problems have never been a large part of the SMT, there is always at least one calculus problem on the written test. During the 2010s, these have been restricted to differentiation and limit problems, but during the 1980s and 1990s there are also integration problems, including volumes of solids of revolution, along with related rates, optimization, and other differentiation problems. They were also more frequent: we find more calculus problems per written test in the 1980s and 1990s than in the 2000s and 2010s.

## 2.4 Emerging Topics

Referring once more to Table 2, We see that in the 2011-2019 period sequences and series emerged as a prominent portion of the SMT. The increase in sequences and series is notable in that over the 1982-1990 period, there were only 15 problems in that category. By 2011-2019, that increased to 66 problems. There was an increase in counting and probability problems as well, from 39 to 67 over those nine-year periods. A similar increase happened in a category not normally associated with high school mathematics: number theory. In 1982-1990, there were 6 problems that could be categorized as number theory; by 2011-2019 there were 44. The type of problem in the 1980s that could be considered “number theory” is the following.

**1989, #9.** The sum of the digits of a certain two-digit number is 12. If we reverse the digits and multiply the number by $4/7$, we get the original number. What is the original number?

**(A)** 48 **(B)** 75 **(C)** 84 **(D)** 93
**(E)** No such number exists

Digit problems such as the preceeding are traditional in math contests. For many years, that was the extent of number theory problems, and one could argue that preceeding problem is not really number theory. As students on math teams learned more about number theory, more number theory became fair game to place in the SMT. But the problems still need to be somewhat accessible to bright students who many not have learned many of these extracurricular topics. Hence, we move from digit problems to more theoretical problems.

**2018, #28.** *Euler's totient function* $\phi(n)$ gives the number of positive integers up to $n$ that are relatively prime to the positive integer $n$, For example, $\phi(4) = 2$ because 1 and 3 are the only positive integers less than 4 that are relatively prime to 4. An interesting property of $\phi$ is

$$\phi(mn) = \phi(m) \cdot \phi(n) \cdot \frac{d}{\phi(d)},$$

where $d = \gcd(m, n)$ is the greatest common divisor of $m$ and $n$. Given that 1009 is prime, find $\phi(9081)$.

**(A)** 3024 **(B)** 4036 **(C)** 6048 **(D)** 6054 **(E)** 12108

One reason for the increase in number theory problems is that number theory forms one of the four broad topics for mathematics olympiads. It is central to the United States of America Mathematics Olympiad (USAMO) and the International Mathematics Olympiad (IMO). Number theory has become one of the foundational topics of contest and olympiad math, even though most of its theory, functions, and concepts are not in the standard high school math curriculum.

The other three topics for mathematical olympiads are algebra, geometry, and the next topic we turn to: combinatorics, or counting problems. These problems have increased greatly on the SMT over the years as well. Indeed the early counting problems are not nearly as challenging as the later ones. When the placement of the counting and probability problems are considered, there is a remarkable difference.

**1985, #49.** In how many ways can seven people stand in a line if two of the people refuse to stand together?

**(A)** 120 **(B)** 1440 **(C)** 6480 **(D)** 3600 **(E)** 5040

That was placed at Number 49, indicating that it is intendend to be difficult. To be fair, it is tricky. However, the next one is a Number 32, and it is certainly more difficult than that Number 49.

**2017, #32.** Suppose we choose up to three given colors to randomly color the sides of a regular pentagon. How many unique such pentagons could be colored where no particular pentagon could be attained by simply rotating another? (That is, we are ignoring reflections, so that — if the colors are $a$, $b$, and $c$ — we count $aaabc$ differently from $aaacb$.)

**(A)** 3 **(B)** 48 **(C)** 51 **(D)** 125 **(E)** 243

At least one problem that could be considered statistics is usually on the SMT written test. But the complexity of the statistics problems has increased commensurate with the increase in students taking AP Statistics and the prominance of statistics in our high school curriculum. The following problem is a typical representative of such "statistics" problems from the 1980s.

**1984, #40.** A set whose mean is 5 contains 7 elements. If each element is multiplied by 10 and then an eighth element equal to 4 is added to the set, what is the mean of the new set?

**(A)** 32.5 **(B)** 50 **(C)** 44.25 **(D)** 9.25 **(E)** 6.75

That same 1984 "statistics" problem on a SMT from the 2010s would probably be placed somehere in the first 10. In fact, the statistics problems on more recent SMTs are actually worthy to be labeled statistics, such as the next problem.

**2006, #44.** The standard deviation, $\sigma$, of a set of data is a measure of "spread" from the mean. It is given by the formula

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n}}$$

where $\mu$ is the mean (average) of the set of numbers, $n$ is the number of data points in the set, and $x_i$ for $i = 1, 2, \ldots, n$ represent the individual data points in the set.

A statistics student taking a test on standard deviation made an unfortunate mistake. Copying a set of 20 numbers, she accidently left off one of the numbers; a 28. Thus, she was only aware of 19 numbers. As a result, her value for $\mu$ was 8 instead of the correct mean: 9. She made no other mistakes and, amazingly, she obtained the standard deviation for the original set! Find the value of this standard deviation.

**(A)** $4\sqrt{38}$ **(B)** 380 **(C)** $2\sqrt{2109}$ **(D)** $2\sqrt{95}$
**(E)** None of these

Interestingly, the problem writer and/or editor apparently thought that not many students would be familiar with standard deviation, and so included the definition!

Another type of problem emerging on the SMT is the base conversion problem, where students are asked to write numbers in binary, base 3, hexadecimal, and so on. The first of these kinds of problems appeared in 1990. They appeared infrequently until 2004. Number base problems have been consistently included ever since.

**2004, #10.** The base 5 representation of a positive integer $N$ is the three-digit integer $abc$, where $a$, $b$, and $c$ are digits. The base 7 representation of the same integer, $N$, is $cba$. What is the remainder when the number $N$ (in base 10) is divided by 17?

**(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Number base problems are not part of the high school mathematics curriculum, but it is traditionally included in computer science courses. It is interesting that the programming/algorithm problems have given way to number base problems.

## 2.5 The Effect of the Graphing Calculator

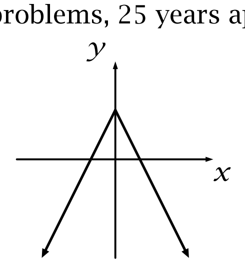
In 1995, the SMT Committee decided that the graphing calculator could be used on the written test. Ever since, students have been able to use a graphing calculator on the test. The implication of this change is that certain types of problems could no longer be asked since the calculator would render them trivial. This resulted in a shift in the type of problems on the written test, from what could be considered formula-driven and computational problems to what could be considered conceptual problems requiring insightful problem-solving. For example, a question like the next one would not appear on any current SMT written test.

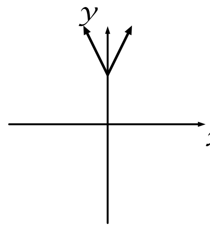
**1984, #24.** Which of these is the polar equation for a line?

**(A)** $\theta = \frac{\pi}{4}$ **(B)** $r = 5$
**(C)** $r = 2\theta$ **(D)** $r = 4\cos\theta$ **(E)** $r = 4\sin\theta$

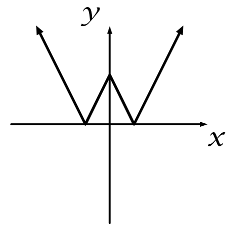
Graphing each answer choice in polar mode on the graphing calculator immediately gives us the answer. Problems concering graphing functions have never really been a large part of the SMT. So the allowance of graphing calculators has not changed the fact that there are so few problems about graphing functions. However, problems about the *graphs of functions* (not *graphing functions*) have always appeared on the SMT. Here are two such problems, 25 years apart.

**1982, #31.** If the figure to the right shows the graph of $y = f(x)$, then which of the following is the graph of $y = |f(x)|$?

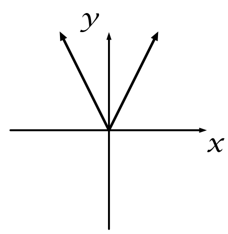




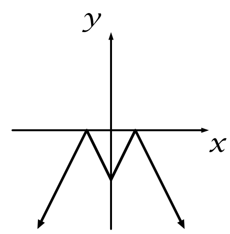

**(A)**




**(B)**



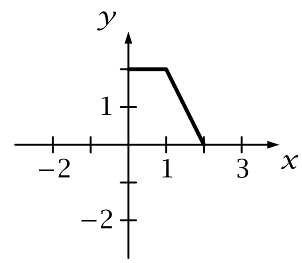

**(C)**

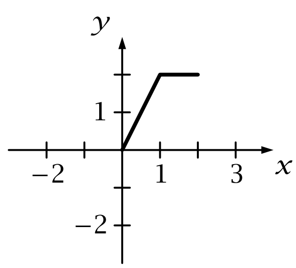



**(D)**

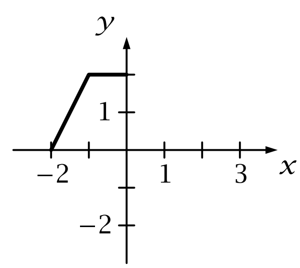
**(E)** None of these

**2017, #4.** To the right is the graph of the function $f(x)$. Which of the following answer choices is the graph of the function $f(2-x)$?

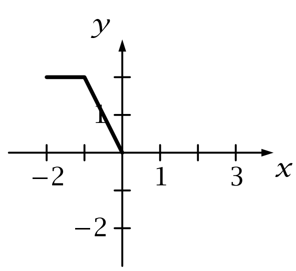


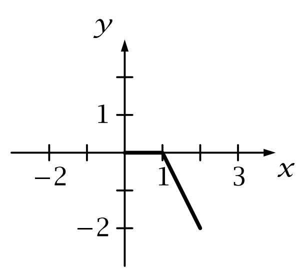

**(A)**




**(B)**

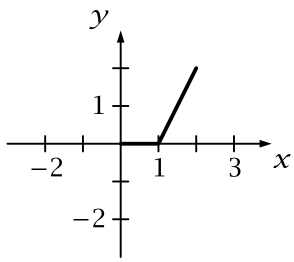



**(C)**




**(D)**

**(E)**




The allowable use of the graphing calculator has dramatically changed the type of trigonometry problems on the SMT. Consider the following trigonometry problem from 1985.

**1985, #12.** For $x$ in its domain, $\cot x - \dfrac{\cos 2x}{\sin x \cos x} = ?$

**(A)** $\sin x$ **(B)** $\cot x$ **(C)** $\sin 2x$ **(D)** $\tan x$ **(E)** $\cos x$

A student is able to answer that problem with a graphing calculator by graphing the expression given in the problem stem, and comparing that to the graph of each answer choice. A similar situation occurs with the next problem, many variations of which have been included in SMTs.

**1985, #46.** If $\tan A = \dfrac{2\sqrt{6}}{5}$ and $0 < A < \dfrac{\pi}{2}$, find $\cos 3A$.

**(A)** $-\dfrac{245}{343}$ and $\dfrac{245}{343}$ **(B)** $-\dfrac{245}{343}$

**(C)** $\dfrac{245}{343}$ **(D)** $-\dfrac{235}{343}$ **(E)** $\dfrac{234}{343}$

In 1985, calculators were not allowed. Thus, the preceeding problem deserved to be placed at Number 46. Without a calculator, that is a time-consuming problem, requiring finesse with trigonometric identities and computations. But with a calculator, that changes dramatically. The student could calculate the inverse tangent of $2\sqrt{6}/5$ in the calculator, obtaining the value of $A$, and then calculate $\cos 3A$ and compare the decimal thus obtained to the fractions. Asking the student to use trigonometric identities to rewrite an expression simply does not occur on the SMT anymore, except on the cipheirng and team rounds where calculators are not allowed. That is not to say that trigonometric identities are not assessed, it is just that the question must be asked differently to limit the use of the calculator. The presence of the calculator means we must find creative ways to challenge students to use trigonometric indentities. This notion extends to Law of Cosines and the Law of Sines.[19] Below is one such problem.

[19]This does not mean that problems calling for calculation of angles using the Law of Sines or the Law of Cosines are absent from the current SMT. Indeed, with the graphing calculator, the concern that the situation works out to give a standard unit circle angle is removed. Thus, a calculator may be *required* to solve these problems.

**2009, #28.** Two scientists, facing each other, stand on perfectly level ground $d$ feet apart. One scientist shines a light beam into the sky at an angle of elevation of $\alpha$ degrees, and the other shines a light beam into the sky at an angle of elevation of $\beta$ degrees. Given that their beams do intersect, how many feet above they ground will they do so?

**(A)** $\dfrac{d\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$ **(B)** $\dfrac{d\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$ **(C)** $\dfrac{d\tan\alpha\tan\beta}{\tan(\alpha-\beta)}$

**(D)** $\dfrac{d\sec\alpha\sec\beta}{\sec(\alpha+\beta)}$ **(E)** None of these

To obtain the correct answer, one must have a good grasp of trigonometric relations. However, it is difficult to develop good problems using, for instance, trigonometric indentities in which the graphing calculator is not an aid. Therefore many of the problems involving trigonometric identities are relegated to the ciphering or team rounds.

Problems involving algebraic manipulation were a popular type of problem on the SMT.

**1982, #41.** Factor completely: $x^2(y^3+1)-2x(y^3+1)-15(y^3+1)$.

**(A)** $(x+5)(x-3)(y+1)(y^2-2y+1)$
**(B)** $(x+5)(x-3)(y-1)(y^2+y+1)$
**(C)** $(y+1)(y^2-y+1)(x-5)(x+3)$
**(D)** $(y^3+1)(x-5)(x+3)$
**(E)** $(y^3+1)(x^2-2x-15)$

But now graphing calculators are allowed. So a student could simply graph the expression in the problem stem, and then graph each answer choice to arrive at the answer. Now, consider the following problem from 18 years later, on which one may use a graphing calculator.

**2000, #20.** To the nearest tenth, find the value of the extraneous root obtained when solving $\sqrt{x}+\sqrt{x+1}-\sqrt{x+2}=0$.

**(A)** $-2$ **(B)** $-2.1$ **(C)** $-2.2$ **(D)** $-2.3$ **(E)** $-2.4$

By asking for the extraneous root, the graphing calculator is no help. The strategy of plugging in the answer choices will not help since none of the answer choices will satisfy the equation. The only choice is to perform the algebra. The preceeding problem is a great example of the creativity of the SMT problem writers: they have provided a calculator-active problem in which the student has no choice but to do some algebraic manipulation.

Complex numbers are another favorite of the SMT. At least one problem involving complex numbers is always included in the SMT each year. However, the calculator renders the type of complex number problem from the 1980s too easy.

**1993, #39.** Simplify $(-1+i)^8$.

**(A)** $-16$ **(B)** $16i$ **(C)** $-16i$ **(D)** $32i$ **(E)** $16$

The following problem from 2019 gets to the same ideas, but in a way where the calculator is not that useful.

**2019, #32.** Let $i = \sqrt{-1}$. The number

$$\left(\frac{-1}{2^{2018/2019}} + i\frac{\sqrt{3}}{2^{2018/2019}}\right)^{2019}$$

can be written in the form $a + bi\sqrt{3}$ where $a$ and $b$ are integers. Compute $a+b$.

**(A)** $-2$ **(B)** $0$ **(C)** $2$ **(D)** $4$ **(E)** $4038$

The allowance of calculators also forces systems of equations problems to be treated a little differently. Here is one from before calculators, presented in matrix notation.

**1986, #3.** If $(x, y)$ is the ordered pair of real numbers satisfying the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix},$$

determine the value of $x - y$.

**(A)** $-19$ **(B)** $1$ **(C)** $3$ **(D)** $5$ **(E)** $13$

In contrast, the next problem is a system of equations problem where the calculator is allowed. Notice this is a Number 1, so it is intended to be the easiest problem on the 50-problem test.

**2007, #1.** Let $a$ be a positive integer. Given the system of equations below, determine the maximum possible value of $x + y + z$.

$$\begin{aligned} 2x + a &= y \\ a + y &= x \\ x + y &= z \end{aligned}$$

**(A)** $-10$ **(B)** $-6$ **(C)** $-\frac{14}{3}$ **(D)** $-2$ **(E)** $0$

## 2.6 Conclusions

The SMT problems have changed over the years, and the change is not entirely due to allowing use of a graphing calculator. Indeed, the graphing calculator seems to have eliminated or greatly reduced the quantity of many rote problems involving algebraic manipulation, trigonometric identities, and graphing functions. However, these skills are still present, but reduced in quantity, and asked in a way that limits the advantage of using a graphing calculator. For instance, a written test problem may still involve factoring a cubic polynomial. However, factoring may only be a necessary step in the problem, and is no longer sufficient on its own to warrant being included on the SMT written test. These kinds of problems still do find a place on the SMT on the ciphering and team rounds, since no calculators are allowed on those rounds. Indeed, on the ciphering and team rounds we can still find problems that mimic the kinds of written test problems from the 1980s.

Other changes include the complete elimination of programming or computer algorithm problems. Yet they have been replaced with problems on number bases, which is another aspect of computer science. No other topic has been eliminated, but some have been reduced in quantity and frequency. Some of these reduced topics are calculus, analytic geometry, and conic sections. For example, while conic sections are always included on the SMT written test, their quantity has been reduced from an average of 4 problems per year in the 1980s to an average of 2 in the 2010s.

The elimination of certain topics was to be expected as other topics emerge: number theory, counting, probability, and statistics, in particular. While number theory and combinatorial problems have dramatically increased—due to the emphasis placed on them at the olympiad level—statistics has not. That statistics questions (besides those involving mean, median, and mode) are included at all is a testament to the proliferation of AP Statistics classes, and the empahsis on statistics in the high school math curriculum. However, they will never form a large proportion of the SMT since statistics, like calculus, is absent at any higher level of competition for high school students.

The SMT written test has become more difficult over the years.[20] There is no question that allowing students to use graphing calculators makes for a more difficult SMT written test, but this is not the only reason. Since the 1980s, it has become easier for the motivated student to learn more mathematics. There are more high school math teams, there are websites like the Art of Problem Solving, and there are more books and materials available specifically for “contest math”. As a result, what was considered an esoteric technique years ago[21] is now

[20] One justification for this assertion, besides the evidence presented of the problems themsleves, is found in the records of the VP for Competitions. In the 1990s, the number of correctly answered problems by the first-place individual was in the mid-40s (out of 50). In the 2010s, the number of correctly answered problems by the first-place individual was in the high 30s or low 40s (out of 50). As a dramatic anecdote of this situation, we offer the 2013 SMT: the first-place student answered 35 correctly, and the second-place student answered 29 correctly. The 2013 SMT also has the distinction of featuring what is the most-correctly answered problem ever (#1, 163 out of 164 correct) and the least-correctly answered problem ever (#45, 0 out of 164 correct).

[21] The counting technique of “stars and bars” or number theortic topics like modular arithmetic

a standard part of a student's toolbox at the SMT. This increases the difficulty of the problems.

Additionally, the fact that the state-wide ARML team is chosen at the SMT is another factor contributing the overall increase in difficulty. The SMT is used as a de-facto "Georgia ARML Team Selection Test." So it is not enough to filter the best teams and individuals to the top, the test (and ciphering) must also help the ARML coaches determine which individuals have demonstrated ability in problem-solving. Thus another layer of filtering is happening. It is not enough for the SMT to be a culmination and celebration of mastery of curricular mathematics, it must also help determine who has problem-solving ability and knowledge of non-curricular mathematics. Thus it is no coincidence that there is increased emphasis on number theory and combinatorics. The students selected for the ARML team need to have demonstrated ability in these topics.

In this article, we presented only 30 of the thousands of SMT problems. There are so many interesting and challenging problems offered at the SMT, I would urge you to obtain some of the SMT problems and give them to the students who like to be challenged, who need enrichment, or who would just like to see what the problems are like. The problems themselves are so well-written and accessible that any student could benefit from working on at least some of the problems. The creativity and dedication of the many people who have written the problems is to be commended.

If the past 42 years of SMTs is anything to go by, the SMT will continue to evolve in response to the curriculum, to the needs of further competition preparation, and to wider accessibility of extracurricular mathematics.

## Appendix

Presented here are the solutions to the problems. They are listed chronologically and then by problem number.

Every extant SMT problem and solution is available in the *State Mathematics Tournament Championship Problem Book* series. There are currently seven volumes available, covering the years 1982 through 2019. (The SMT problems from 1977 to 1981 are missing.) Volume 1 is available for free, and there are links to purchase the others, on the GCTM website.

**1982, #31.** Taking the absolute value of a function keeps any part of the graph lying above $x$-axis where it is, and reflects about the $x$-axis any part below the $x$-axis. This must be the graph in B.

**1982, #41.** We see that the expression factors as

$$\begin{aligned}(y^3+1)(x^2-2x-15) &= (y^3+1)(x-5)(x+3)\\ &= (y+1)(y^2-y+1)(x-5)(x+3),\end{aligned}$$

are two examples.

so the answer is C.

**1982, #43.** By Power of a Point, we have $AC \cdot BC = CD^2$. Then $4AC = 64$ so that $AC = 16$. Thus $AB = AC - BC = 16 - 4 = 12$. It follows that $AB$ is the same length as the radius of the circle. Hence, $\triangle AOB$ is equilateral, and its area is $\frac{\sqrt{3}}{4} \cdot 12^2 = 36\sqrt{3}$. The area of sector $AOB$ is one-sxith of the area of the circle: $\frac{1}{6} \cdot \pi \cdot 12^2 = 24\pi$. Finally, the area of the shaded segment of the circle is $24\pi - 36\sqrt{3}$. The answer is D.

**1983, #26.** We trace through the program in the following table.

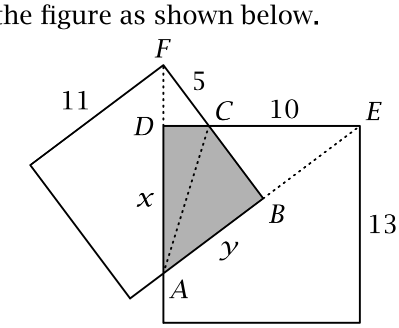
| $N$ | $A$ | $B$ |
|---|---|---|
| | 2 | 1 |
| 1 | 2 | 1 |
| 2 | 2 | 2 |
| 3 | 4 | 2 |
| 4 | 8 | 4 |
| 5 | 32 | 8 |

It follows that the number printed is 8. The answer is C.

**1983, #30.** The equations are those of a circle centered at the origin and an upward-facing parabola. There will be two intersections, so the answer is C.

**1984, #24.** Answer choice A, $\theta = \frac{\pi}{4}$, is a line through the origin of slope 1. The other answer choices are, in order: a circle of radius 5; a spiral; a circle of radius 2; and a circle of radius 2.

**1984, #27.** We label the figure as shown below.




Drawing $\overline{AC}$, we see that the lozenge is simply two right triangles on a common hypotenuse. Adding the areas of the two right triangles gives the area of the lozenge. Call $AD = x$ and $AB = y$. Since $DE = 13$ and $CE = 10$, then $DC = 3$. Likewise, $BF = 11$ and $CF = 5$, so $BC = 6$. Now we have some relationships given by the Pythagorean Theorem on $\triangle ADC$ and $\triangle ABC$: $x^2 + 9 = AC^2$ and $y^2 + 36 = AC^2$. These imply that $y^2 = x^2 - 27$.

Notice that $\triangle ABF$ is also right, and that $DF = \sqrt{CF^2 - DC^2} = \sqrt{25-9} = 4$. Then we have $BF^2 + AB^2 = AF^2$, or $121 + y^2 = (x+4)^2$. Substituting for $y^2$, we have $121 + x^2 - 27 = x^2 + 8x + 16$. This simplifies to $94 = 8x + 16$, which gives $x = \frac{39}{4}$. Hence, $y = \sqrt{(\frac{39}{4})^2 - 27} = \sqrt{\frac{1089}{16}} = \frac{33}{4}$.

Finally, the area is

$$\frac{1}{2} \cdot \frac{33}{4} \cdot 6 + \frac{1}{2} \cdot \frac{39}{4} \cdot 3 = \frac{99}{4} + \frac{117}{8} = \frac{315}{8}.$$

The answer is A.

**1984, #40.** The sum of the elements in the set originally is $5 \cdot 7 = 35$. Multiplying each of these by 10 results in a sum multiplied by 10; so the sum after multiplying by 10 is 350. Adding 4 gives 354. Dividing this by 8 (the new number of elements) gives a mean of $\frac{354}{8} = \frac{177}{4} = 44\frac{1}{4} = 44.25$.

**1985, #12.** We write everything in terms of $\cos x$ and $\sin x$. We have

$$\begin{aligned} \cot x - \frac{\cos 2x}{\sin x \cos x} &= \frac{\cos x}{\sin x} - \frac{2\cos^2 x - 1}{\sin x \cos x} \\ &= \frac{\cos^2 x - 2\cos^2 x + 1}{\sin x \cos x} \\ &= \frac{1 - \cos^2 x}{\sin x \cos x} \\ &= \frac{\sin^2 x}{\sin x \cos x} \\ &= \frac{\sin x}{\cos x} = \tan x. \end{aligned}$$

The answer is D.

**1985, #46.** First, from the fact that $\tan A = \frac{2\sqrt{6}}{5}$, we may label angle $A$ in a right triangle and label the sides opposite and adjacent as $2\sqrt{6}$ and 5, respectively. By the Pythagorean Theorem, the hypotenuse is $\sqrt{(2\sqrt{6})^2 + 5^2} = \sqrt{24+25} = 7$. Thus, $\sin A = \frac{2\sqrt{6}}{7}$ and $\cos A = \frac{5}{7}$. Now we find a way to express $\cos 3A$ in terms of $\cos A$ and $\sin A$. We have

$$\begin{aligned} \cos 3A &= \cos(A + 2A) \\ &= \cos A \cos 2A - \sin A \sin 2A \\ &= \cos A(2\cos^2 A - 1) - \sin A(2 \sin A \cos A) \\ &= \frac{5}{7}\left(2 \cdot \frac{25}{49} - 1\right) - \frac{2\sqrt{6}}{7}\left(2 \cdot \frac{2\sqrt{6}}{7} \cdot \frac{5}{7}\right) \\ &= \frac{5}{7} \cdot \frac{1}{49} - \frac{2\sqrt{6}}{7} \cdot \frac{20\sqrt{6}}{49} \\ &= \frac{5}{343} - \frac{240}{343} = -\frac{235}{343}. \end{aligned}$$

**1985, #49.** We use complementary counting. We will count the number of ways the seven people may stand without restriction, and then subtract from that the number of ways they may stand if the two people must stand together. The number of ways seven people may stand in a line is 7!. Stick the two people together to create one "super-person" and then arrange the 6 "people". This can be done in $2 \cdot 6!$ ways (since there are 2 ways the two people may stand together). Hence, the number of ways they may stand in a line if the two people refuse to stand next to each other is $7! - 2 \cdot 6! = 6!(7-2) = 6! \cdot 5 = 720 \cdot 5 = 3600$. The answer is D.

**1986, #3.** The given matrix equation is equivalent to the system of equations $2x+3y = 0$ and $3x+y = 7$. The first equation implies that $y = -\frac{2}{3}x$. Substituting this into the second equation yields $3x - \frac{2}{3}x = 7$ so that $x = 3$. Then $y = -2$ and $x - y = 3 - (-2) = 5$, and the answer is D.

**1987, #15.** Rationalizing the denominator leads to

$$\begin{aligned}\frac{\left(\sqrt{x+1}+\sqrt{x-1}\right)^2}{x+1-(x-1)} &= \frac{x+1+2\sqrt{(x+1)(x-1)}+x-1}{2}\\ &= x+\sqrt{x^2-1} = 3.\end{aligned}$$

Then we have

$$\begin{aligned}\sqrt{x^2-1} &= 3-x\\ x^2-1 &= 9-6x+x^2\\ 6x &= 10\\ x &= \frac{5}{3}.\end{aligned}$$

The answer is D.

**1989, #6.** We have

$$11 * 0 = \frac{11^2-0}{11-0} = \frac{121}{11} = 11$$

and

$$5 * x = \frac{5^2-x}{5-x} = \frac{25-x}{5-x}.$$

Hence,

$$\frac{25-x}{5-x} = 11$$

implies $25 - x = 55 - 11x$. Thus, $10x = 30$ so that $x = 3$ and the answer is E.

**1989, #9.** We require a two-digit number whose digit sum is 12 and is divisible by 7. This limits the possibilities to the single option 84. Four-sevenths of 84 is 48. The answer is A.

**1993, #39.** First Solution. Note that $(-1+i)^2 = 1 - 2i + i^2 = -2i$. Then $(1+i)^8 = (-2i)^4 = 16i^4 = 16$. Second Solution. We use DeMoivre's Theorem. Since $-1+i = \sqrt{2}e^{3\pi i/4}$, we compute $(-1+i)^8 = \left(\sqrt{2}e^{3\pi i/4}\right)^8 = 16e^{6\pi i} = 16$. The answer is E.

**2000, #20.** We solve the equation.

$$\begin{aligned}\sqrt{x} + \sqrt{x+1} &= \sqrt{x+2} \\ x + 2\sqrt{x(x+1)} + x + 1 &= x + 2 \\ 2\sqrt{x(x+1)} &= 1 - x \\ 4x(x+1) &= 1 - 2x + x^2 \\ 3x^2 + 6x - 1 &= 0\end{aligned}$$

By the quadratic formula, the roots are $-1 \pm \frac{2}{3}\sqrt{3}$. Of course the extraneous root is the negative root, which is approximately $-2.2$. The answer is C.

**2004, #10.** We are given $N = 25a + 5b + c = 49c + 7b + a$, which implies $12a - b - 24c = 0$. So $12a = b + 24c$, but since $a$, $b$, and $c$ are positive integers, this forces $b = 0$ so that $a = 2c$. Then the number $N$ is base 10 is $50c + c = 51c$. Hence all such numbers are multiples of 51, and since 51 divided by 17 leaves no remainder, the answer is zero, which is A.

**2006, #44.** We have that the two standard deviation expressions, one for 20 numbers and the other for 19 numbers, are equal. Hence,

$$\begin{aligned}\frac{\sum_{i=1}^{20}(x_i - 9)^2}{20} &= \frac{\sum_{j=1}^{19}(x_j - 8)^2}{19} \\ 19\sum_{i=1}^{20}(x_i^2 - 18x_i + 81) &= 20\sum_{j=1}^{19}(x_j^2 - 16x_j + 64).\end{aligned}$$

Distributing the summation yields

$$19\left(\sum_{i=1}^{20} x_i^2 - 18\sum_{i=1}^{20} x_i + \sum_{i=1}^{20} 81\right) = 20\left(\sum_{j=1}^{19} x_j^2 - 16\sum_{j=1}^{19} x_j + \sum_{j=1}^{19} 64\right).$$

Evaluating the latter two sums in each expression in parentheses gives us

$$19\left(\sum_{i=1}^{20} x_i^2 - 18(180) + 1620\right) = 20\left(\sum_{j=1}^{19} x_j^2 - 16(152) + 1216\right).$$

Therefore, we obtain

$$\begin{aligned} 19\left(\sum_{i=1}^{20} x_i^2 - 1620\right) &= 20\left(\sum_{j=1}^{19} x_j^2 - 1216\right) \\ 19\sum_{i=1}^{20} x_i^2 - 30780 &= 20\sum_{j=1}^{19} x_j^2 - 24320 \\ 19\sum_{i=1}^{20} x_i^2 &= 20\sum_{j=1}^{19} x_j^2 + 6460. \end{aligned}$$

Now, subtract $19\sum_{j=1}^{19} x_j^2$ from both sides. Thus, since $\sum_{i=1}^{20} x_i^2 - \sum_{j=1}^{19} x_j^2 = (28)^2 = 784$, we obtain

$$\begin{aligned} 19\left(\sum_{i=1}^{20} x_i^2 - \sum_{j=1}^{19} x_j^2\right) &= \sum_{j=1}^{19} x_j^2 + 6460 \\ 19(784) &= \sum_{j=1}^{19} x_j^2 + 6460 \\ \sum_{j=1}^{19} x_j^2 &= 8436. \end{aligned}$$

Therefore,

$$\sigma = \sqrt{\frac{\sum_{j=1}^{19} x_j^2 - 1216}{19}} = \sqrt{\frac{8436 - 1216}{19}} = \sqrt{380} = 2\sqrt{95}.$$

Thus, the answer is D.

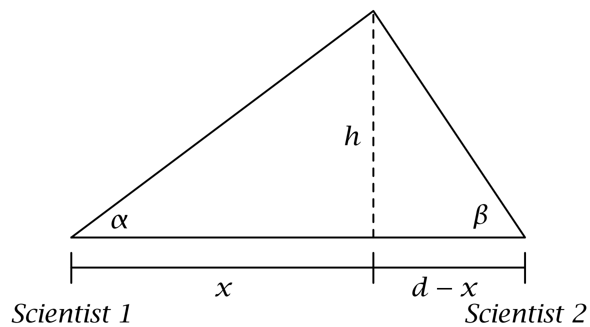
**2007, #1.** The second equation implies $y = x - a$; substituting this into the first yields $2x + a = x - a$, or $x = -2a$. Hence, $y = -3a$, and $z = x + y = -5a$. The sum is therefore $x + y + z = -10a$. For the sum to be a maximum, we set $a = 1$ (since $a$ must be a positive integer) to get $-10$. The answer is A.

**2007, #20.** We expand each binomial to get

$$(1 - 6x + 12x^2 - 8x^3)(1 + 2kx + k^2x^2).$$

From here, it is easily seen that the coefficient of the $x^2$ term will be $k^2 - 12k + 12$. The sum of the two solutions is 12. The answer is D.

**2009, #28.** We have the following diagram.




Note that the angle at the top (where the light beams meet) is supplementary to $\alpha+\beta$. Thus, the sine of the angle at the top is $\sin(\alpha+\beta)$. By the Law of Sines, the length of the side opposite angle $\alpha$ is $\frac{d\sin\alpha}{\sin(\alpha+\beta)}$ and the length of the side opposite $\beta$ is $\frac{d\sin\beta}{\sin(\alpha+\beta)}$. Since the area of the triangle is given by the formula $\frac{1}{2}ab\sin\gamma$, we have that the area is

$$A = \frac{1}{2}\cdot\frac{d\sin\alpha}{\sin(\alpha+\beta)}\cdot\frac{d\sin\beta}{\sin(\alpha+\beta)}\cdot\sin(\alpha+\beta) = \frac{d^2\sin\alpha\sin\beta}{2\sin(\alpha+\beta)}.$$

But the area is also given by one-half of the base times the height; that formula gives $\frac{1}{2}dh$. Equating the results of the two formulas allows us to cancel $\frac{1}{2}d$ from both sides, resulting in

$$h = \frac{d\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

The answer is A.

**2010, #13.** We have that $1 \bowtie 2 = \frac{5}{4}$, and that $\frac{5}{4} \bowtie 3 = \frac{169}{120}$. Then $\frac{169}{120} \bowtie z = \frac{5}{4}$ gives

$$\frac{14400z^2 + 28561}{40560z} = \frac{5}{4},$$

or

$$57600z^2 + 114244 - 202800z = 0.$$

This factors into $4(60z-169)(240z-169) = 0$, so the solutions are $\frac{169}{60}$ and $\frac{169}{240}$. The positive difference is then $\frac{169}{60} - \frac{169}{240} = \frac{676-169}{240} = \frac{507}{240} = \frac{169}{80}$. Thus, the answer is B.

**2011, #35.** The foci of the hyperbola lie at the points $F_1(7,3)$ and $F_2(-3,3)$. The distance from $P$ to $F_1$ is $\frac{21}{4} - 3 = \frac{9}{4}$. The distance from $P$ to $F_2$ is

$$\begin{aligned}\sqrt{(7-(-3))^2 + \left(\frac{21}{4}-3\right)^2} &= \sqrt{10^2 + \left(\frac{9}{4}\right)^2}\\ &= \sqrt{100 + \frac{81}{16}}\\ &= \sqrt{\frac{1681}{16}} = \frac{41}{4}.\end{aligned}$$

Therefore, the total distance is $\frac{9}{4} + \frac{41}{4} = \frac{50}{4} = \frac{25}{2}$. The answer is D.

**2016, #24.** Tangent segments from the same point are congruent, so $AE = 12 = AB$. Furthermore, $\triangle AEC \sim \triangle ADE$, so

$$\begin{aligned} \frac{12}{AC+10} &= \frac{AC}{12} \\ 12 \cdot 12 &= AC(AC+10) \\ 144 &= AC^2 + 10AC \\ AC^2 + 10AC - 144 &= 0 \\ (AC+18)(AC-8) &= 0, \end{aligned}$$

which implies $AC = 8$, since $AC$ cannot be negative. Therefore $AC \cdot AB = 8 \cdot 12 = 96$, and the answer is D.

**2017, #4.** Note that when $x = 1$ we have $f(2-1) = f(1) = 2$. So the function $f(x)$ and $f(2-x)$ both must contain the point $(1, 2)$. The only graph of the answer choices which contains $(1, 2)$ is A. (Note that, in general, replacing $x$ with $h - x$ in $f(x)$, for real $h$, creates a reflection over the line $x = h/2$.)

**2017, #32.** First Solution. Ignoring rotations, $3^5$ pentagons can be drawn with the five sides in three colors. However, 3 of these have sides which are entirely in one color. Thus, $3^5 - 3 = 240$ pentagons have more than one color and fall into sets of 5 pentagons that appear distinct when considering orientation but could be rotated to obtain one another. Hence, there are $240/5 = 48$ distinctly colored pentagons with more than one color when considering rotations. This leads to $48 + 3 = 51$ distinctly colored pentagons so the answer is C.

Second Solution. There are three cases: only one color, using two colors, or using three colors.

*Case I: One color.* If only one color $A$ is used, the arrangement must be of the form $AAAAA$, and there are three possibilities of colors. Hence, there are 3 ways with only one color.

*Case II: Two colors.* If two colors $A$ and $B$ are used, the arrangement must be of the form $AABAA$, or $ABBAA$, or $ABABA$. For each of these 3 arrangements, there are 3 ways to choose the first color and 2 ways to choose the second color. Thus there are $3 \cdot 3 \cdot 2 = 18$ possibilities.

*Case III: Three colors.* If all three colors $A$, $B$, and $C$ are used, the arrangement must be of the form $AAABC$, or $AABAC$, or $ABCAB$, or $AABCB$, or $AABCC$. For each of these 5 arrangements, there are 3 ways to choose the first color, 2 ways to choose the second color, and only one way to choose the third color. Thus there are $5 \cdot 3 \cdot 2 = 30$ possibilities.

Finally, the total number of ways is $3 + 18 + 30 = 51$. Thus, the answer is C.

**2018, #28.** Note that $9081 = 9 \cdot 1009$, $\gcd(9, 1009) = 1$, and $\phi(1) = 1$. So, we have

$$\phi(9081) = \phi(9) \cdot \phi(1009) \cdot \frac{1}{1}.$$

Since 1009 is prime, every positive integer less than 1009 is relatively prime to 1009. Hence $\phi(1009) = 1008$. In general if $p$ is prime, $\phi(p) = p - 1$. Moreover, $9 = 3 \cdot 3$, $\gcd(3,3) = 3$, and $\phi(3) = 2$ because 3 is prime. Hence $\phi(9) = \phi(3) \cdot \phi(3) \cdot \frac{3}{\phi(3)} = 6$.

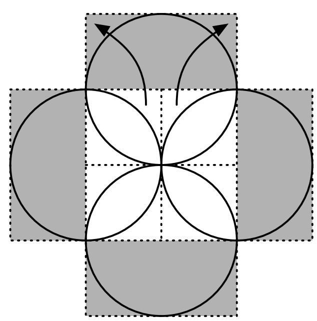
Therefore, $\phi(9081) = \phi(9) \cdot \phi(1009) = 6 \cdot 1008 = 6048$, and the answer is C.

**2019, #32.** We use DeMoivre's Theorem. Since $-1 + i\sqrt{3} = 2e^{2i\pi/3}$, we have

$$\begin{aligned}\left(\frac{-1}{2^{2018/2019}} + i\frac{\sqrt{3}}{2^{2018/2019}}\right)^{2019} &= \left(\frac{1}{2^{2018/2019}} \cdot 2e^{2i\pi/3}\right)^{2019} \\ &= \frac{1}{2^{2018}} \cdot 2^{2019} \cdot e^{4038i\pi/3} \\ &= 2e^{1346i\pi} = 2.\end{aligned}$$

Hence, $a = 2$ and $b = 0$ so that $a + b = 2$ and the answer is C.

**2019, #34.** Draw diameters of the circles in such a way so that they form a square enclosing the non-shaded regions. Then the portions of the square which are shaded can be moved to the outer semicircles to enclose those semicircles with rectangles. Hence, the area of the shaded region is exactly the sum of the areas of the four rectangles.



Each rectangle has dimensions equal to the diameter and radius of the circles, so the area $3.25 \times 6.5$, and there are four of them. Therefore the total area is $4 \times 3.25 \times 6.5 = 84.5$. Thus the answer is B.