The House of Wisdom

The History of Mathematics, Part 12

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Outline

Algebraic Advancements

Clever Counting

Geometric Genius

Trigonometric Talent

Homework

Islamic Mathematics

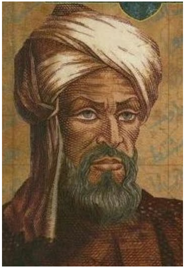
- Islam arose as a monotheistic religion and political force under Muhammad, c. 630
- By 732, Islamic culture spread around the Mediterranean, from Turkey to North Africa to Spain (and east to India)
- Baghdad became capitol in 766
- c. 800, Caliph al-Mansur established a royal library in Baghdad: Bayt al-Hikma (House of Wisdom)

Islamic Mathematics

- Need to keep existing laws led to philosophical background of those laws, which led to other Greek and Babylonain works
- There were things understood by “the ancients” not understood at this time
- Translations of many Greek works in Arabic
- Overall, enthralled by Greek proofs but not constrained by them
- Extended much of Greek mathematics and created new

Outline

al-Khwārizmī



Abu Ja'far Muhammad ibn Musa al-Khwārizmī

780-850

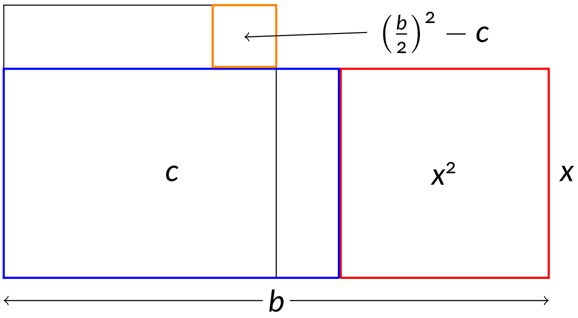
"That fondness for science, that affability and condescension which God shows to the learned, that promptitude with which he protects and supports them in the elucidation of obscurities and in the removal of difficulties, has encouraged me to compose a short work on calculating."

al-Khwārizmī

- Wrote *Al-Kitāb al-mukhtasar fī hisāb al-jabr wa-l-muqābala* in 830 (*The Compendious Book on Calculation by Completion and Balancing*)
- Words *algebra* and *algorithm* derived from him and his work
- Considered the father of algebra, although justifications were geometric

al-Khwārizmī

The solution to $x^2 + c = bx$ is $x = \frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - c}$.



al-Khwārizmī

His work has many problems to practice his methods.

"I have divided 10 into two parts; I have multiplied the one by 10 and the other by itself, and the products were the same."

al-Khwārizmī

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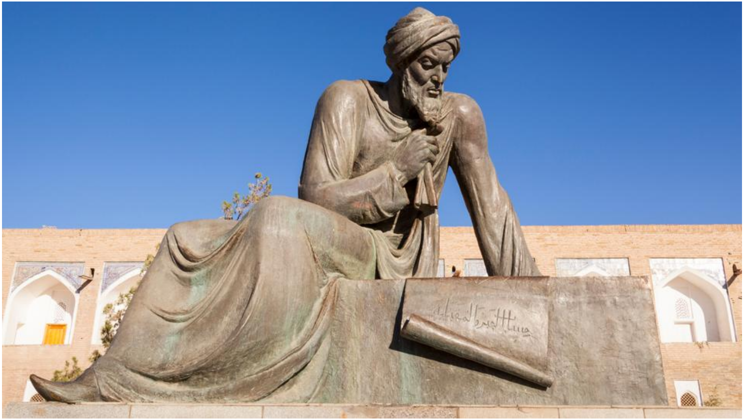
"I have divided 10 into two parts; I have multiplied the one by 10 and the other by itself, and the products were the same."

The equation is $10x = (10 - x)^2$ and the solution is $x = 15 - \sqrt{125}$. The positive root $15 + \sqrt{125}$ is rejected because it is too large to be "part" of 10.

al-Khwārizmī



Al-Khwarizmi monument in Khiva, Uzbekistan

After al-Khwārizmī

- Fifty years later, geometric foundations were based on Euclid
- But they were still concerned with numerical examples
- Implication: an ease with irrationals!
- One solution to a problem in the work of Abū Kāmil c.900 has

$$x = 10 + \sqrt{50} - \sqrt{50 + \sqrt{20,000} - \sqrt{5000}}.$$

Outline

al-Haytham

Abu Ali al-Hasan ibn al-Haytham

965-1040

"Given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer."

al-Haytham

- Optics: we see becuase light *enters* the eye
- Introduced proof by induction
- Used induction to prove formulas for sums of integers:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \left(\frac{n}{3} + \frac{1}{3}\right) n \left(n + \frac{1}{2}\right),$$

and

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n}{4} + \frac{1}{4}\right) n(n+1)n.$$

Counting

Ibn Yahya al-Maghribi Al-Samaw'al (1130-1180)

- Used binomial coefficients
- Explained laws of exponents
- Gave another proof of al-Haytham's formulas

Aḥmad al-Ab'dari ibn Mun'im (d.1228)

- Answered old question of the number of possible words that could be formed from letters in Arabic alphabet.
- Used binomial coefficients to represent the number of ways to count something
- Started combinatorics

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Omar Khayyam

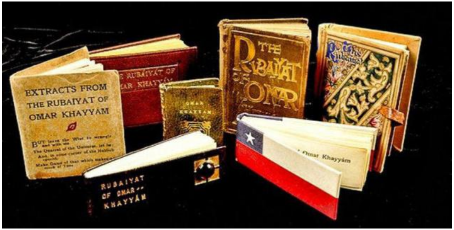
ʻUmar ibn Ibrāhīm al-Khayyāmī (Omar Khayyam)

1048-1131

"Whether at Naishapur or Babylon,/ Whether the Cup with sweet or bitter run,/ The Wine of Life keeps oozing drop by drop,/ The Leaves of Life keep falling one by one."

Omar Khayyam

- Reformed the calendar (length of year accurate to 6 places)
- Critical look at Euclid's 5th postulate
- Solved every type of cubic geometrically
- Many minor scientific and mathematical discoveries
- Most famous for his poetry

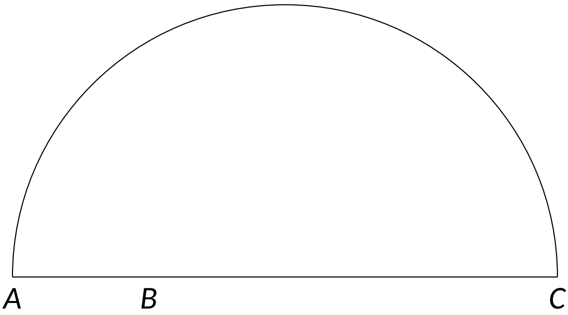


Editions of *The Rubaiyat*

Khayyam's Cubic Solution

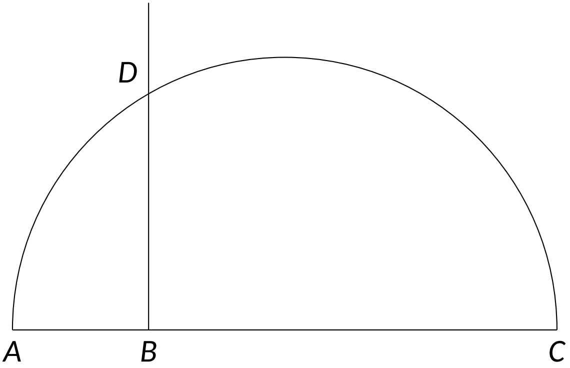
We solve $x^3 + b^2x + a^3 = cx^2$ by Khayyam's method.

First, set $AB = \frac{a^3}{b^2}$ and $BC = c$ and draw $\overline{AC}$. Then draw a semicircle with diameter $\overline{AC}$.



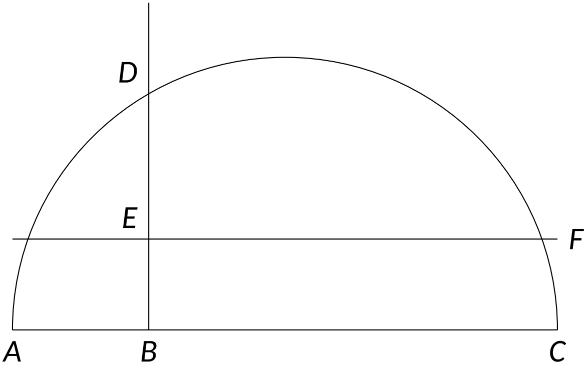
Khayyam's Cubic Solution

Construct perpendicular to $\overline{AC}$ at B so that it intersects the semicircle at D.



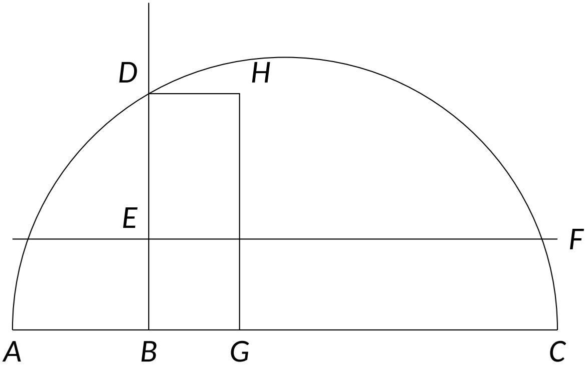
Khayyam's Cubic Solution

On $\overline{BD}$ mark off BE so that its length is b. Then draw $\overline{EF}$ parallel to $\overline{ED}$.



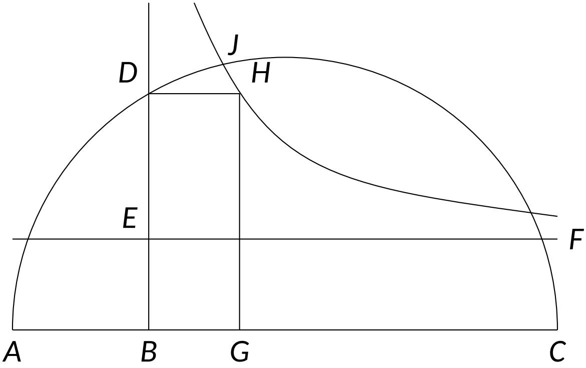
Khayyam's Cubic Solution

Mark G on $\overline{BC}$ such that $ED : BE = AB : BG$. Draw rectangle $DBGH$.



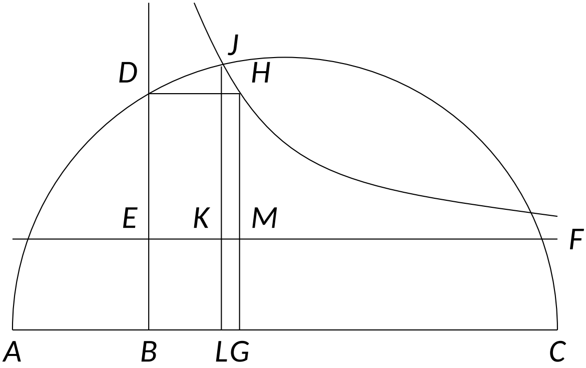
Khayyam's Cubic Solution

Through H, draw the hyperbola whose asymptotes are $\overline{EF}$ and $\overline{ED}$. The hyperbola intersects the semicircle at J.



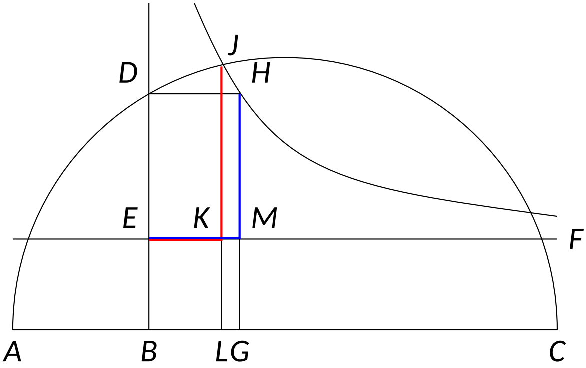
Khayyam's Cubic Solution

Through J, draw parallel to $\overline{DE}$. This parallel intersects $\overline{EF}$ at K and $\overline{AC}$ at L. Finally, $\overline{GH}$ and $\overline{EF}$ intersect at M.



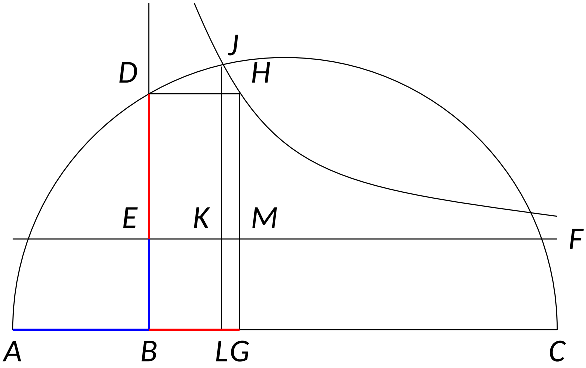
Khayyam's Cubic Solution

Since J and H are on the hyperbola, $EK \cdot KJ = EM \cdot MH$.



Khayyam's Cubic Solution

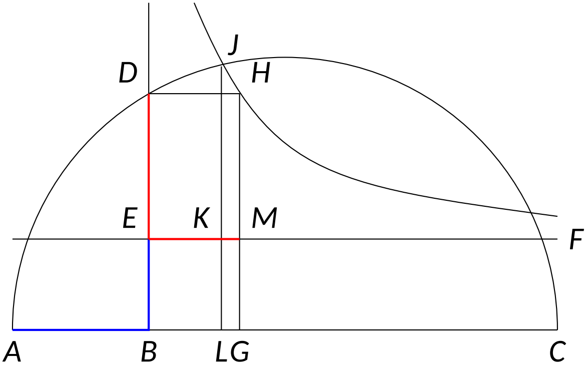
By construction, $ED : BE = AB : BG$, so $BG \cdot ED = BE \cdot AB$.



Khayyam's Cubic Solution

Therefore,

$$EK \cdot KJ = EM \cdot MH = BG \cdot ED = BE \cdot AB.$$



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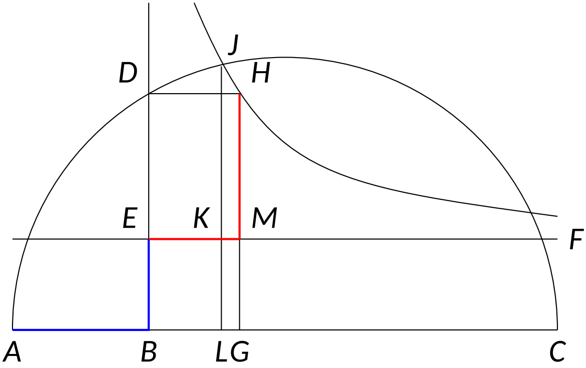
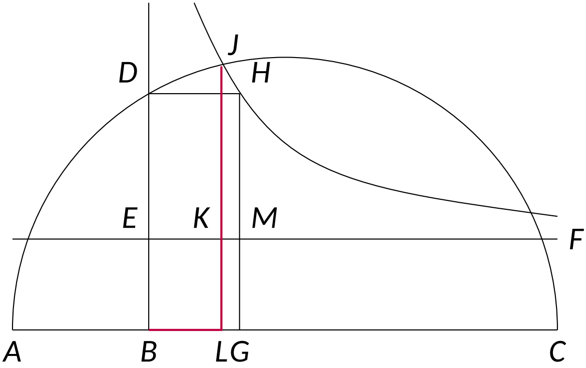


Khayyam's Cubic Solution

Now, $BL \cdot LJ = EK(BE + KJ) = EK \cdot BE + EK \cdot KJ = EK \cdot BE + BE \cdot AB = BE(EK + AB) = BE \cdot AL$.
Thus, $BL^2 \cdot LJ^2 = BE^2 \cdot AL^2$.



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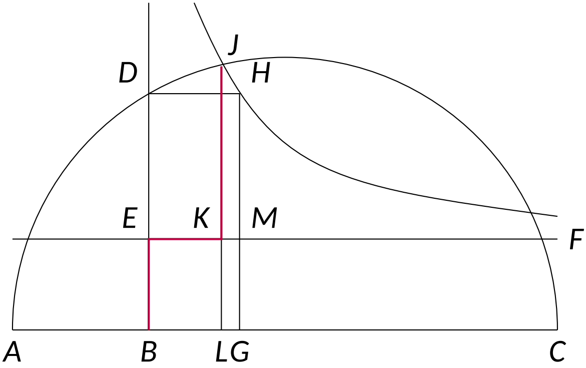

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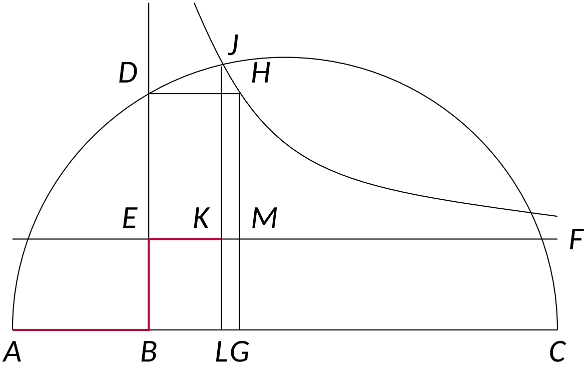

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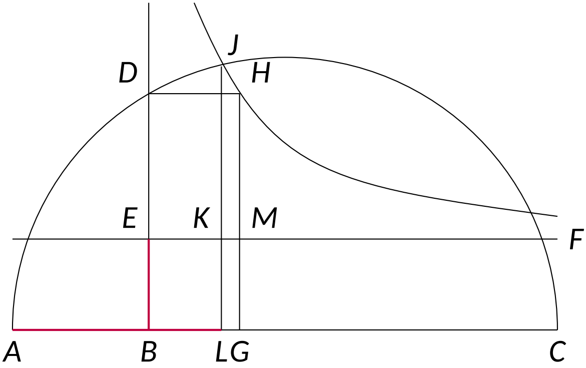
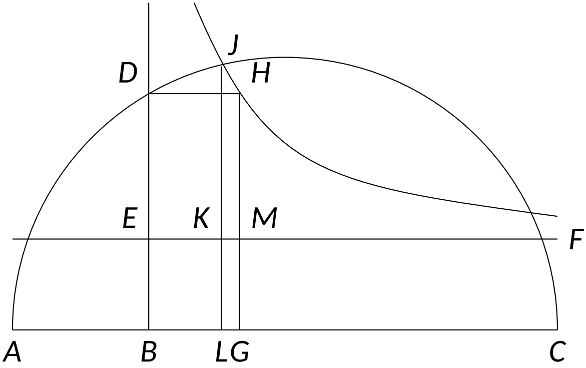


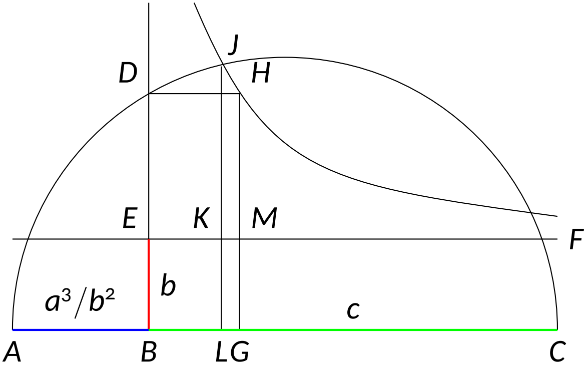
Khayyam's Cubic Solution

Note $\overline{LJ}$ is a perpendicular in a semicircle, so we have $LJ^2 = AL \cdot LC$. Thus, $BL^2 \cdot LC = BE^2 \cdot AL$, or
$BE^2(BL + AB) = BL^2(BC - BL)$.



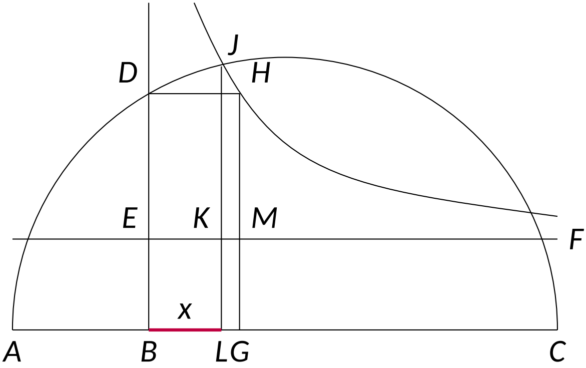
Khayyam's Cubic Solution

Recall $BE = b$, $AB = a^3/b^2$, and $BC = c$.
Then $BE^2(BL + AB) = BL^2(BC - BL)$ becomes
$b^2(BL + a^3/b^2) = BL^2(c - BL)$.



Khayyam’s Cubic Solution

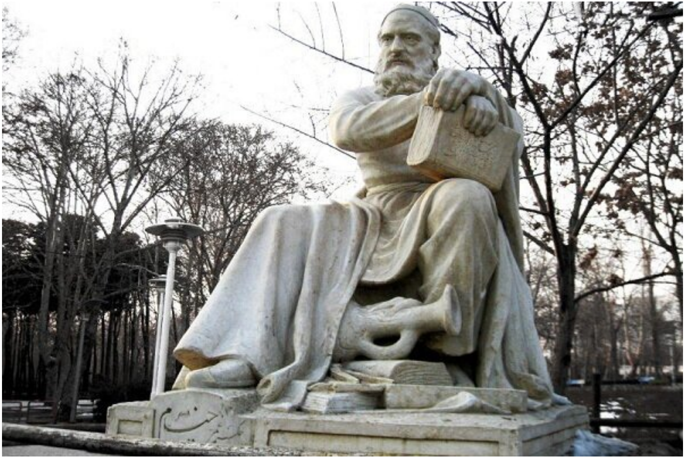
Expanding and rearranging gives
$BL^3 + b^2BL + a^3 = cBL^2$. Hence BL is the root of the cubic!



Omar Khayyam

Khayyam's tomb in Nishapur, Iran

Omar Khayyam



Khayyam monument in Nishapur, Iran
May 17 is National Khayyam Day in Iran

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Trigonometry

Using Ptolemy and Indian astronomy, Islamic scholars advanced trigonometry.

al-Battānī

Abū 'Abdallah Muḥammad ibn Jābir al-Battānī

855-929

al-Battānī

- First use of “cosine”; no negatives, so only valid for angles up to 90°.
- For obtuse angles, used “versine”: $\text{versin}(\alpha) = R + R\sin(\alpha - 90^\circ)$.
- Refined the length of the year, to 365 days 5 hours 46 minutes 24 seconds

al-Bīrūnī

Abu al-Rāyḥan Muḥammad ibn Aḥmad al-Bīrūnī

973-1055

"Once a sage asked why scholars always flock to the doors of the rich, whilst the rich are not inclined to call at the doors of scholars. "The scholars" he answered, "are well aware of the use of money, but the rich are ignorant of the nobility of science"."

al-Bīrūnī

- Completely discussed all six trig ratios and their use
- Calculated trigonometric tables out to four sexagesimal places

al-Kāshī

Ghiyāth al-Dīn Jamshīd al-Kāshī

1380-1429

al-Kāshī

- First complete mastery of decimals
- Although used sexagesimal places in his spherical trig table
- Calculated $\sin(1^\circ)$ to 9 sexigesimal places accurately (equivalent to 18 decimal places)

Ulūgh Beg

Ulūgh Beg

1394-1449

"It is the duty of every true Muslim, man and woman, to strive after knowledge."

Ulūgh Beg

- al-Kāshī's patron
- Ruler of Samarkand (now Uzbekistan)
- Astronomer and mathematician
- Established schools, scientific meetings
- Built the largest and best observatory in the east

Ulūgh Beg's Observatory

Built in 1425.

Outline

Homework

- Translating the first algebra;
  Math Through the Ages, Sketch 10

Next: The Blockhead