Last-Minute Problems, No. 4

1 TEXTBOOK PROBLEMS. [2] p.163 #1; p.164 #4; p.169 #4(a); p.170 #5.

2 THE EUCLIDEAN ALGORITHM. [2] There is non-geometrical material in the *Elements*, such as the Euclidean algorithm, a process for finding the greatest common divisor (gcd) of two numbers. This is found at the beginning of Book VII, but was presumably known before Euclid's time. The process is this:

> Divide the larger of the two numbers by the smaller one. Then divide the divisor by the remainder. Continue this process, of dividing the last divisor by the last remainder, until there is no remainder. The final divisor is the gcd.

For example, let us find the gcd of 126 and 210. We divide 210 by 126 to get 1 with remainder 84. Next, we divide the divisor, 126, by the remainder, 84, to get 1 with new remainder 42. Now divide the previous divisor, 84, by the previous remainder, 42, to get 2 with no remainder. Now that we have no remainder, we know that the gcd was the last nonzero remainder we used: 42. So the gcd of 126 and 210 is 42. **2a)** Find the gcd of 481 and 851. **2b)** Find the gcd of 5913 and 7592. **2c)** Find the gcd of 1827 and 3248. In all three parts, your work must reflect appropriate use of the Euclidean Algorithm.

3 THERE ARE INFINITELY MANY PRIMES. [5] From Euclid's Prop. IX-20, we know that we'll never run out of primes.** In addition, since all primes but 2 are odd, we can divide the odd primes into two categories—those that are one more than a multiple of 4 (e.g., 5, 13, 17, 29, …) and those that are three more than a multiple of 4 (e.g., 3, 7, 11, 19, …). Obviously, at least one of these two categories of primes must be infinite. In what follows, we'll modify Euclid's proof of IX-20 to show that there are infinitely many primes of the form $p = 4n + 3$.

3a) Prove that the product of two numbers, each of which is one more than a multiple of 4, is itself one more than a multiple of 4. In other words, if $a = 4m + 1$ and $b = 4n + 1$, then ab also has this form. (Of course, all numbers here are positive integers.)

3b) Now suppose that $\{a, b, c, d, \dots, w\}$ is a finite collection of primes (in Euclid's words, an "assigned multitude"), each having the form $4n+3$. We introduce a new number $M = 4(abcd \cdots w) - 1$. Mimic Euclid's two cases from IX-20 to show that there must be a prime of the form $4n + 3$ not among the original multitude. Conclude that there are infinitely many primes of the form $4n + 3$. (*Hint:* Use the property that positive integers have a unique prime factorization and part 3a.)

3c) Are there infinitely many *composites* of the form $4n + 3$? Explain.

**If you haven't read IX-20 yet, you really should before starting this problem. For this and other problems, you may find it helpful to consult the home of the amazing on-line version of the *Elements* at `https://mathcs.clarku.edu/~djoyce/java/elements/elements.html`

$\boxed{4}$ THE LOGICAL ROLE OF THE PARALLEL POSTULATE. [2] None of the propositions in Book I prior to Proposition I-29 uses the Parallel Postulate in its proof, whereas all the later results in Book I depend on the Parallel Postulate, with a single exception. Find it. Speculate as to why Euclid didn't put it before I-29.

$\boxed{5}$ THE PYTHAGOREAN THEOREM. . . ONE MORE TIME. [2] The previous six proofs of the Pythagorean Theorem from Last-Minute Problems #2 give an array of possible ways of establishing the theorem, but it should be noted that Euclid used none of them in the *Elements*. Read the proof contained in the *Elements* (Prop. I-47). Which of the previous six could he have used as Prop. I-47? Do you give Euclid high marks for the proof he actually devised for the *Elements*, or did he miss an easy proof?

$\boxed{6}$ THE PLATONIC SOLIDS—THEY SWEAR THEY ARE JUST FRIENDS. [2.5] The final book of the *Elements* is devoted to the construction of the regular polyhedra known as the Platonic Solids: the tetrahedron (all 4 faces are equilateral triangles), cube (6 squares), octahedron (8 equilateral triangles), dodecahedron (12 regular pentagons), and icosahedron (20 equilateral triangles). The first three solids were known before the Greeks, and there is an extant bronze dodecahedron dating from the seventh century BC. The icosahedron was apparently first studied by Theaeteus (417-369 BC), who also proved that these are the only regular polyhedra. Euclid constructed these by inscribing them in spheres; then compared the exact side lengths of each solid to the diameter of the sphere. If we let the diameter of the sphere be 1 unit, these are the side lengths found by Euclid:

$$\text{Tetrahedron: } \sqrt{\frac{2}{3}} \qquad \text{Cube: } \sqrt{\frac{1}{3}} \qquad \text{Octahedron: } \sqrt{\frac{1}{2}}$$

$$\text{Dodecahedron: } \frac{\sqrt{15}-\sqrt{3}}{6} \qquad \text{Icosahedron: } \frac{\sqrt{50-10\sqrt{5}}}{10}$$

What is truly remarkable is that, with the lack of a good number system and the complete absence of algebraic notation, Euclid accomplished this task! Your task is to find decimal approximations to these measurements, then to find the (approximate) surface area of each solid. (You may want to look up the surface area formulas.)

$\boxed{7}$ A SMATTERING OF "ELEMENTAL" RESULTS. [7.5] Using your online resources, look up each of the Propositions mentioned in the problems below.

7a) What familiar relationship is expressed in Prop. I-47? How can you determine what that relationship is just from the diagram?

7b) Read the proof of Prop. II-1. Translate this into a simple algebraic identity.

7c) Read the proof of Prop. II-13. What is the famous result from trigonometry that you should recognize here?

7d) Read the proof of Prop. III-1. Use a compass and a straightedge to follow along with the proof, and construct the center of a given circle yourself. What method of proof did Euclid use?

7e) Read the statement of Prop. III-16. What type of line is Euclid defining here?

7f) What does Prop. XII-2 say in modern language? That is, what formula is implied by this Proposition?

7g) Translate Prop. XII-7, Prop. XII-10, and Prop. XII-18 into modern algebraic formulas.

7h) Look up online the logo of the national high school mathematics honor society, Mu Alpha Theta. Do you recognize that particular arrangement of shapes? Where does it come from?

$\boxed{8}$ DATA. [1.5] One of Euclid's other works was called simply *Data*. A "datum" is a set of parts or relations of a figure such that if all but any one is given, then the remaining one is determined. For example, angle A, opposite side a, and circumradius R of a triangle constitute a datum, since given any two the third is determined (in this case, by the relation $a = 2R \sin A$). Show that the following parts of a triangle constitutes a datum by finding the formula or property that relates the three quantities: **8a)** the angles A, B, C; **8b)** angle A, side b, altitude h_c on side c; **8c)** the ratios of the sides: a/b, b/c, c/a.

$\boxed{9}$ PUSHING PAST GEOMETRICAL LIMITS. [3]

9a) Knowing how to measure lengths of straight-line segments, how might one define the length of the circumference of a circle?

9b) Knowing how to measure areas of polygons, how might one define the area of a circle?

9c) Let P_n and A_n denote the perimeter and the area of a regular n-gon circumscribed about a circle of unit diameter. Find

$$\lim_{n\to\infty} P_n \quad \text{and} \quad \lim_{n\to\infty} A_n.$$

9d) Knowing the volume of a pyramid is given by one-third the area of its base times its altitude, how might one arrive at the formula for the volume of a circular cone?

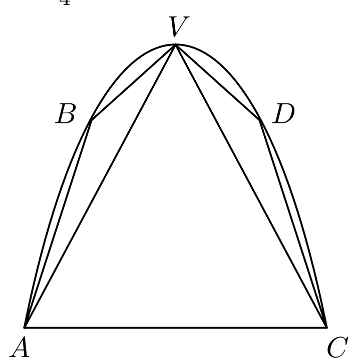
We know that the area of the lateral surface of a regular prism (the rectangular faces that constitute the sides) is given by the perimeter of its base times its altitude. One might think that the lateral surface area of a circular cylinder could be defined as the limit of the areas of any sequence of inscribed polyhedral faces, provided the number of faces indefinitely increases so that the area of each face approaches zero. Mathematicians were surprised when, in the early 1860s, H.A. Schwartz (1843-1921) showed that this is not so. Schwartz's example was so astonishing at the time that it became known as *Schwartz's paradox*.

$\boxed{10}$ THE QUADRATURE OF THE PARABOLA. [3] One of Archimedes' most remarkable accomplishments was in determining the area of a parabolic segment. Let AVC be a parabolic segment with vertex V and base AC. Archimedes then proceeds to inscribe triangles. The first triangle is $\triangle AVC$. In the two portions left over, he inscribes two triangles, $\triangle ABV$ and $\triangle CDV$; in the four smaller portions left over, he inscribes four triangles, and so on—completely filling the parabola with triangles. Archimedes then calculated that the total areas of the triangles at each stage was $\frac{1}{4}$ the area of the triangles from the previous stage. Thus, after n stages, the sum of the areas of all inscribed triangles is equal to

$$\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^n}\right) \triangle AVC.$$

As n increases indefinitely, all triangles equal the parabolic segment. Hence, since the sum of the "infinitely many" fractions is an infinite geometric series, we have that the area of the parabolic segment is equal to

$$\frac{1}{1 - \frac{1}{4}} \triangle AVC = \frac{4}{3} \triangle AVC.$$



Thus, the area of any parabolic segment is $\frac{4}{3}$ the area of the inscribed triangle!

For example, to find the area of the segment formed by $y = 2 - x^2$ and the x-axis, we inscribe a triangle so that the base lies on the x-axis and one vertex is on the y-axis (like in the picture above—imagine $\overline{AC}$ is the x-axis and the y-axis goes through V). Then the base of the triangle is the distance between the x-intercepts of $y = 2 - x^2$; the x-intercepts are $-\sqrt{2}$ and $\sqrt{2}$, so the base has length $\sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}$. The height of the triangle is the y-intercept of $y = 2 - x^2$; this is 2. Therefore the area of the inscribed triangle is $(1/2) \cdot 2 \cdot 2\sqrt{2} = 2\sqrt{2}$. Thus, the area of the parabolic region is $(4/3) \cdot 2\sqrt{2} = 8\sqrt{2}/3$.

Without using calculus, find the area of the region between: **10a)** $y = 4 - x^2$ and the x-axis; **10b)** $y = 10 - x^2$ and the x-axis; **10c)** $y = 16 - x^2$ and the x-axis.

$\boxed{11}$ ARCHIMEDES, AN APPROXIMATING $\beta\alpha\delta\alpha\sigma\sigma$. [1.5] In the midst of his approximation for the value of π, Archimedes needed a value for $\sqrt{3}$ and he used

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}.$$

How good is this as a decimal?

12 YOU, AN APPROXIMATING $\beta\alpha\delta\alpha\sigma\sigma$. [4] An interesting question raised in the previous problem is how one would get such a sharp approximation for square roots without benefit of a calculator. A very nice algorithm is the following.

To approximate $\sqrt{A}$, begin with an initial approximation of x_0 (obtained by "eye-balling it"). Then let the next approximation be

$$x_1 = \frac{x_0^2 + A}{2x_0},$$

then the next one is

$$x_2 = \frac{x_1^2 + A}{2x_1}$$

and generally, use the recursive definition

$$x_{n+1} = \frac{x_n^2 + A}{2x_n}. \tag{1}$$

12a) Assuming that the sequence of successive approximations converges to a limit L, show that $L = \sqrt{A}$. (*Hint:* Take limits of both sides of Eq. (1) as $n \to \infty$.)

12b) Now suppose we want to approximate $\sqrt{3}$ and we start with the rational number $x_0 = 5/3$ (this is reasonable since $(5/3)^2 = 25/9 \approx 27/9 = 3$). Apply the recursion formula twice to approximate this square root.

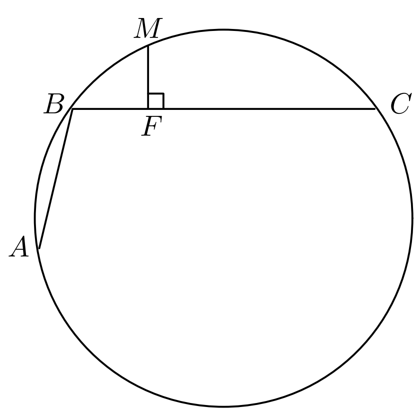
12c) Notice anything? Do you think Archimedes was on to something?

12d) Using two iterations of this recursive procedure, approximate: $\sqrt{2}$ with $x_0 = 3/2$; $\sqrt{5}$ with $x_0 = 2$; $\sqrt{56}$ with $x_0 = 7.5$; $\sqrt{90}$ with $x_0 = 9.5$; and $\sqrt{2519}$ with x_0 of your choice. (If these seem familiar, that's because these are the square roots you were asked to approximate in Problem 3 from Last-Minute Problems #1.)

13 BROKEN CHORD. [4] Islamic scholars have attributed to Archimedes the *Theorem of the Broken Chord*, which says

> If AB and BC make up a broken chord in a circle, where $BC > AB$, and if M is the midpoint of arc ABC, the foot F of the perpendicular from M on BC is the midpoint of the broken chord ABC.

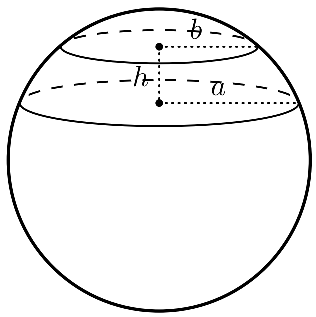
Prove this theorem.



14 SPHERICAL SEGMENT. [4.5] Archimedes reputedly considered volumes of certain spherical parts, such as the following: Two planes, both parallel to the plane that contains a great circle, intersect a sphere. The resulting "slice" of the sphere is called a *spherical segment of two bases*. The volume of the segment is

$$V = \frac{\pi h(3a^2 + 3b^2 + h^2)}{6}$$

where a is the radius of the lower base, b is the radius of the upper base, and h is the altitude between the two bases.



14a) Show that the above formula is equivalent to the sum of a sphere of radius $h/2$ and two circular cylinders whose altitudes are each $h/2$ and whose radii are a and b.

14b) Find the formula for the volume of a *spherical segment of one base* by considering what happens in the above formula as b approaches zero.

14c) Find the formula for the volume of a hemisphere using the above formula. (*Hint:* As b approaches zero, and as a approaches r, the radius of the sphere, what happens to h?)

14d) If possible, derive the volume of a sphere from the above formula.

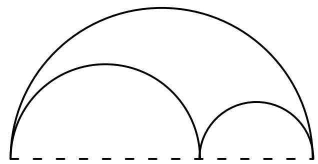
15 ARCHIMEDEAN "FORMULAS" IN GEOMETRIC DISGUISE. [3] In his masterpiece *On the Sphere and Cylinder*, Archimedes stated his results about spherical volume and area by comparing his figures with such better-understood figures as cylinders and cones. Assuming we know the modern formulas for the key properties of cones and cylinders, translate the following statements into familiar, modern-day formulas.

15a) "Any sphere is equal (by volume) to four times the cone which has its base equal to the greatest circle in the sphere and its height equal to the radius of the sphere."

15b) "Every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere is half again as large as the sphere."

15c) "Every cylinder whose base is the greatest circle in a sphere and whose height is equal to the diameter of the sphere has surface (together with its bases) that is half again as large as the surface of the sphere."

16 THE QUADRATURE OF THE ARBELOS. [6] Let A, C, and B be three collinear points, with C between A and B. Semicircles having AC, BC, and AB as diameters are drawn on the same side of the line. The figure bounded by the three semicircles is called an *arbelos*. The term "arbelos" means "shoemaker's knife" in Greek, and this term is applied to the figure which resembles the blade of a knife used by ancient cobblers. Once again, it was Archimedes who investigated the fascinating properties of this figure.



16a) At C, erect a perpendicular to AB that intersects semicircle AB at G. Let the common external tangent to the two smaller semicircles touch these semicircles at T and W. You will prove that GC and TW are equal.

- **a1:** Draw $\overline{TW}$ and connect the radii of the two smaller circles to $\overline{TW}$. Then construct a segment parallel to $\overline{TW}$ from the center of the smallest semicircle. This creates a right triangle. Find the sides of the right triangle in terms of AC and BC.
- **a2:** Show that TW is equal to the length of the longest leg of the right triangle, and find a simplified expression for TW in terms of AC and BC.
- **a3:** Find an expression for GC in terms of AC and BC, and conclude that $TW = GC$. (*Hint:* Use Problem 8c from Last-Minute Problems #3.)

16b) Prove that the area of the arbelos equals the area of the circle that has GC as a diameter.