Order From Chaos

The History of Mathematics, Part 6

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Outline

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Oneness

Is the universe one, or made of independent entities?

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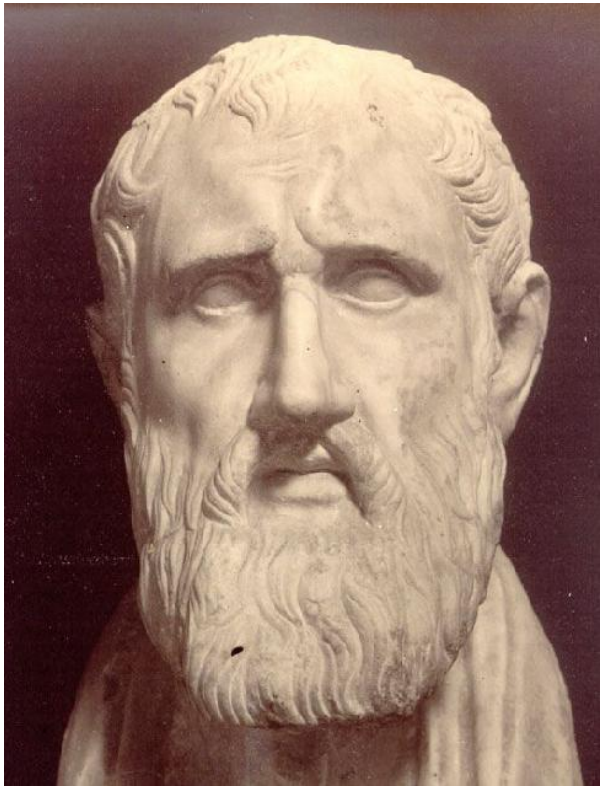
Parmenides (c.500 BC-450 BC) developed these answers:

- *Monism* claims that the diversity of objects are a single external reality
- This reality is *Being*
- "All is One"
- "Non-Being" is impossible

This is the "One vs. Many" argument.
Parmenides' student, Zeno, argued for the One by trying to contradict the Many.

Zeno



Zeno of Elea

490 BC-430 BC

"If being is many, it must be both like and unlike, and this is impossible, for neither can the like be unlike, nor the unlike like."

Zeno's Paradoxes

- Zeno believed that reality is unchanging and sense impressions merely illusions.
- Showed that current ideas on motion required careful criticism to avoid logical paradoxes.
- Zeno's Plan: Support the One by showing that the Many would lead to inconsistencies.

Zeno's Paradoxes

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- Showed that current ideas on motion required careful criticism to avoid logical paradoxes.
- Zeno's Plan: Support the One by showing that the Many would lead to inconsistencies.
  - If there are many things then how many are they?
  - How big are they?
  - Do they make a noise?
  - Where are they?
  - How can they move?

Zeno's Paradoxes

By taking his opponents' premises and reducing them to absurdity, Zeno developed four paradoxes which must be resolved in any coherent theory. The premises are:

Premise 1 Space and time are infinitely divisible.

Premise 2 Space and time are made up of indivisible atoms.

Zeno's Paradoxes: The Dichotomy

Dichotomy: motion is impossible, because before an object can travel any given distance, it must first travel half the distance; but before it does this it must first travel half of this, and so on.

Zeno's Paradoxes: Achilles

Achilles: Achilles is racing a tortoise, who starts ahead. Before Achilles can pass the tortoise, he must first reach the point P_1 where the tortoise started. Say he does this when the tortoise is at P_2. Before Achilles passes the tortoise, he must first reach P_2, and so on.

Zeno's Paradoxes: The Arrow

The Arrow: Motion is impossible, because at any atomic instant, the arrow is at rest in space; if not, space would be infinitely divisible. At the next instant, it is somewhere else at rest. So it is always at rest.

Zeno's Paradoxes: The Stadium

The Stadium: Two chariots A and B race round the stadium at the same speed but in opposite directions. A third chariot G is at rest. Suppose at some atomic instant, B racing left passes a unit length of G. Then in the same time, A and B pass two unit lengths of each other. But then they pass one unit in half the time, which is indivisible.

Zeno's responses in more detail.

Zeno's Paradoxes

Implications:

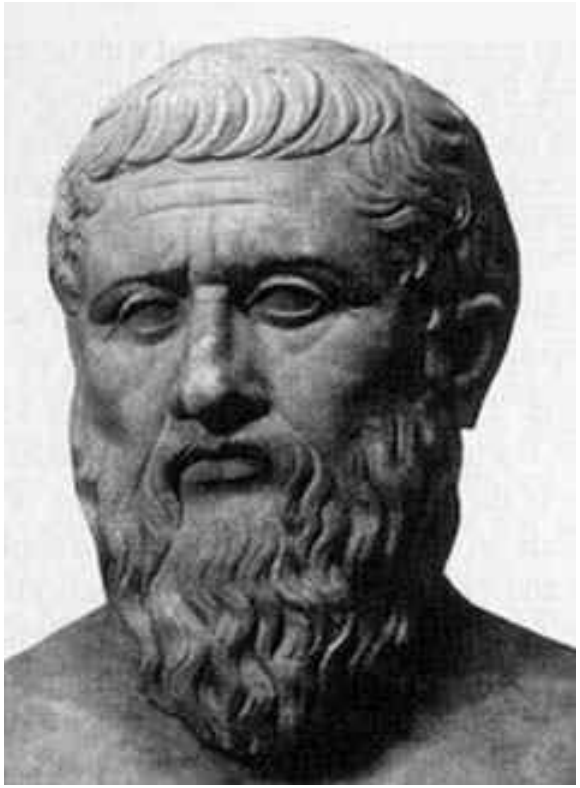
- Distinction between actual and potential infinity
- Distinction between number and magnitude
- What is the continuous? The discrete?
- How do we model time and motion?
- How many points are on a line segment?
- How many fractions are there between 0 and 1?
- How do we measure the perimeter of an island?

Zeno


MY CLIENT COULDN'T HAVE KILLED ANYONE WITH THIS ARROW, AND I CAN PROVE IT!
I'D LIKE TO EXAMINE YOUR PROOF, ZENO. YOU MAY APPROACH THE BENCH.
—BUT NEVER REACH IT!

Outline

Plato



Plato

427 BC-347 BC

"He is unworthy of the name of man who is ignorant of the fact that the diagonal of a square is incommensurable with its side."

Plato

Important for three reasons

1. Philosopher, not primarily a mathematician, but greatly advanced mathematics (geometry in particular)
2. His works are the best source of info on mathematics during this time
3. Arguably the greatest influence on thought and culture

Plato

- Founded a school in Athens in 387 BC
  in a part of Athens called *Academy*
- Above the entrance: “Let no one ingorant of geometry enter here”
- Wrote books, taught and lectured hundreds of students
- In 367 BC a 17-year old came to Academy…

Aristotle

Aristotle

384 BC-322 BC

"I count him braver who overcomes his desires than him who conquers his enemies, for the hardest victory is over self."

Aristotle

- Wrote extensively, and disagreed with Plato on many topics
- Became teacher/tutor to Prince Alexander of Macedon
- Alexander supported Aristotle's new school in Athens, the Lyceum
- Through Alexander's conquests, spread Aristotle's ideas east and brought back ideas from other cultures

Aristotle

- Codified logical thought into syllogisms, including
  - Postulates (truths particular to that science)
  - Axioms (truths common to all)
- Statements cannot be both true and false

Aristotle

- Codified logical thought into syllogisms, including
  - Postulates (truths particular to that science)
  - Axioms (truths common to all)
- Statements cannot be both true and false
- To define a thing means to establish its existence
- Quantity consists of two categories:
  - Number is discrete
  - Magnitude is continuous

Aristotle

- Wanted to refute Zeno, but failed to convincingly prove properties of infinite sets. Result:
  - Rejected the *actual* infinity; accepted only *potential* infinity
  - For instance, any line can be doubled; given any set of points, another can always be found
- Aristotle's influence changed the Greek definition of the word *mathematikos*:
  - from "that which can be known" or "any kind of study or learning"
  - to a particular kind of high-quality knowledge; the most important knowledge
- Aristotle's views on science persisted for 2000 years

Outline

Logical Discourse

Logical Discourse: a sequence of statements obtained by deductive reasoning from an accepted set of initial statements

- Define terms of the discourse
- Set down primary statements whose truths are accepted
- All other terms defined by previous terms
- All other statements logically deduced from previously accepted statements

Example of a Discourse

Definitions

Person any man, woman, child in the collection S

Club any nonempty subset of S

Conjugate clubs two clubs having no members in common

Postulates

P1 Every person of S is a member of at least one club

P2 For every pair of persons of S there is exactly one club to which both belong

P3 For every club there is exactly one conjugate club

Example of a Discourse

Theorem (T1)

Every person of S is a member of at least two clubs.

Proof.

Suppose a is a member of S. By P1 there is a club A to which a belongs. By P3 there exists a club B conjugate to A. Since B is nonempty, it has at least one member, b, and $b \neq a$. By P2 there exists a club C containing a and b. Since A and B are conjugates, b is not in A, implying $A \neq C$. Thus a belongs to two clubs, A and C. □

Example of a Discourse

Theorem (T2)

Every club contains at least two members.

Proof.

Let A be a club. Since A is nonempty, it has at least one member a. Suppose a is the only member of A. By T1, there is a club B different from A and containing a. Noe B must contain a second member, for otherwise A and B would not be distinct. By P3, there is a club C such that B is conjugate to C. Thus A is also conjugate to C. But this contradicts P3. Hence, there must be two members of A. □

Example of a Discourse

Theorem (T3)

S contains at least four persons.

Proof.

In the proof of T1, we established the existence in *S* of at least two different persons *a* and *b*. By P2, there is a club *A* to which *a* and *b* both belong. By P3 there is a club *B* conjugate to *A*. But by T2, *B* must contain at least two members, *c* and *d*. Since *A* and *B* are conjugate, *a*, *b*, *c*, and *d* must be distinct. □

Why is This Important?

- Before Plato, a proof was more like *corroboration* of what seemed likely
- After Aristotle, a statement is true only because there is proof
  (i.e., that two segments are incommensurable does not “seem” likely; it requires proof)
- A proof is the discovery of a truth

Why is This Important?

- Discover the unknown from the known
- Ancient Greeks thought this was the only way to learn
- Much of Greek thought was organized in this manner: mathematics, medicine, law, science, etc

Evolutionary Vs. Revolutionary

Evolutionary Deductive process gradually developed over time (Pythagoras)

Revolutionary Deductive process was created in response to some crucial circumstance (Eudoxus)

Outline

Homework

- Last-Minute Problems, #3 – due February 22
- The Platonic Solids
  Math Through the Ages, Sketch 15

Next: The Mathematician's Bible