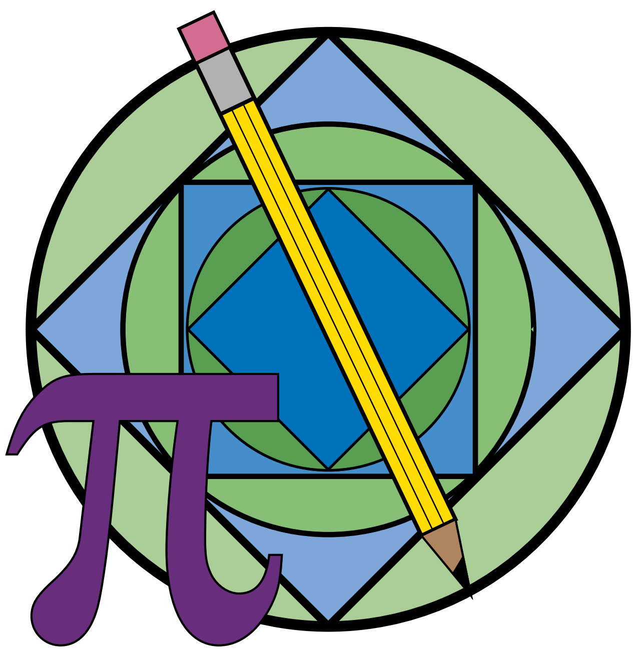
Mastering Competition Mathematics
Volume 1: Functions and Polynomials



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Mastering Competition Mathematics
Volume 1: Functions and Polynomials

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Dedicated to the memory of Dr. Otis Jackson "Jay" Cliett, III (1944-2017).

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PREFACE

> There is one story about [Hilbert's] attitude toward literature which also reveals a great deal about his feeling for mathematics. It seems that there was a mathematician who had become a novelist. "Why did he do that?" people in Göttingen marveled. "How can a man who was a mathematician write novels?" "But that is completely simple," Hilbert said. "He did not have enough imagination for mathematics, but he had enough for novels."
>
> — *Constance Reid,* Hilbert

The purpose of this book is provide high school students with practice solving typical math competition problems as found on actual high school competitions. All the problems contained in this book are sourced from mathematics tournaments which have occurred in the state of Georgia, USA.

Each chapter begins with some expository material with examples. The expository material attempts to explain to the reader some basic and important mathematical problem-solving techniques, illustrated by a variety of examples. Following some examples are exercises. The exercises give the reader a chance to test their understanding immediately. The exercises are numbered consecutively with the problems. The examples and exercises are competition problems, but specifically curated to help the reader understand a topic they may be learning for the first time.

The problem sections in each chapter are divided into two parts, "JV" (for "Junior Varsity") and "Varsity". In Georgia, nearly every tournament is offered in these divisions. The JV division is typically for students who have not yet enrolled in a precalculus course while Varsity is for any K-12 student. Many JV problems are easier than the Varsity problems, but that does not mean that every JV problem is easy! I have attempted to put the JV problems and Varsity problems in order of difficulty, but that is always a subjective measure.

The problems are divided into JV and Varsity to provide mathematical challenges for students as they move through high school. Middle and high school students using this book can use the JV problems to master the topic while in middle school or the early years of high school. Then the student can tackle the Varsity problems afterwards. Even the exposition can and should be re-read. The first reading does not always suffice to master a topic; multiple readings are warranted.

"To reread a book is to read a different book. The reader is different. The meaning is different." — Johnny Rich

As a result of this dual purpose to support younger and older students, there are many problems where the same concept appears. Usually, problem books

Part I

THE PROBLEMS

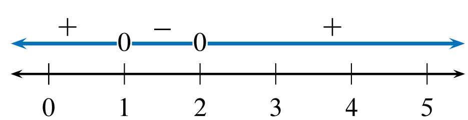
5 INEQUALITIES

Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty's sake and pulls it down to earth.

— *Marston Morse*

The Basics

The vast majority of inequality problems appearing on high school math competitions are the kind of problems one learns to solve in math class. Problems such as to find the solution set to $x^2 - 3x + 2 \le 0$. To solve such a problem, we rely on a notion from Chapter 2 on absolute value: the *critical number*. Recall that the critical number is the value that makes the expression zero. In the case of $x^2 - 3x + 2 \le 0$, the critical numbers are $x = 1$ and $x = 2$. The critical numbers split the real number line into three sections — $x < 1$, $1 < x < 2$, and $x > 2$ — where the quantity $x^2 - 3x + 2$ will either be positive or negative. We can visualize this using a number line, where we put above the number line a 0 where $x^2 - 3x + 2$ is zero, "+" representing positive intervals, and "–" representing negative intervals. We can determine the intervals by choosing any number in the interval and evaluating $x^2 - 3x + 2$.



So we can see that $x^2 - 3x + 2 \le 0$ for $1 \le x \le 2$.

With more complicated expressions, it will be necessary to combine like terms or add/subtract fractions in order to determine the critical numbers.

Example 1. Solve $x^3 - 3x^2 - 6x + 8 < 0$. [GAC 2003, Algebra II]

(A) $2 < x < 4$ **(B)** $x < -2, 1 < x < 4$ **(C)** $-2 < x < 1, x < 4$
(D) $x < -2, x > 4$ **(E)** None of these

Solution. To find the critical numbers, we factor the expression x^3-3x^2-6x+8. This factors into $(x+2)(x-1)(x-4)$, so the critical numbers are $x = -2$, $x = 1$, and $x = 4$. They split the reals into the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 4)$, and $(4, \infty)$. Now we check whether $f(x) = x^3 - 3x^2 - 6x + 8 = (x+2)(x-1)(x-4)$ is positive or negative by evaluating f at convenient values within each interval.

$$\begin{aligned} f(-3) = -1 \cdot -4 \cdot -7 < 0 &\implies f \text{ is negative for } x < -2. \\ f(0) = 2 \cdot -1 \cdot -4 > 0 &\implies f \text{ is positive for } -2 < x < 1. \\ f(2) = 4 \cdot 1 \cdot -2 < 0 &\implies f \text{ is negative for } 1 < x < 4. \\ f(5) = 7 \cdot 4 \cdot 1 > 0 &\implies f \text{ is positive for } x > 4. \end{aligned}$$

Therefore, the solution to $x^3 - 3x^2 - 6x + 8 < 0$ is $x < -2$ or $1 < x < 4$. Thus, the answer is **B**. ◆

Example 2. Solve $\dfrac{x+1}{x^2+4x+3} \geq 0$. [Cobb 2005, Varsity]

(A) $(\infty, -3)$ **(B)** $(-\infty, -3]$ **(C)** $[-3, -1] \cup [-1, \infty)$
(D) $(-3, -1) \cup (-1, \infty)$ **(E)** $(-3, \infty)$

Solution. We have a rational function in this problem. The critical numbers are the numbers that make both numerator and denominator zero. Factoring yields $f(x) = (x+1)/((x+3)(x+1)) = 1/(x+3)$. The critical numbers are $x = -3$ and $x = -1$, so there are three intervals we must investigate: $(-\infty, -3)$, $(-3, -1)$, and $(-1, \infty)$. We calculate:

$$\begin{aligned} f(-4) = \frac{1}{-1} < 0 &\implies f \text{ is negative for } x < -3. \\ f(-2) = \frac{1}{1} > 0 &\implies f \text{ is positive for } -3 < x < -1. \\ f(0) = \frac{1}{3} > 0 &\implies f \text{ is positive for } x > -1. \end{aligned}$$

Note that $1/(x+3) \neq 0$ for any x so no interval can be closed. Therefore, the solution to the inequality is $(-3, -1) \cup (-1, \infty)$. Thus, the answer is **D**. ◆

The following is different kind of inequality problem, where we must determine an integer that satisfies the inequality rather than an interval of solution.

Example 3. Which of the following is the least positive integer n for which $n^{24} > 14^{16}$? [GSW 2012]

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Solution. The powers are big, and testing each answer choice is extremely tedious. But we can deal with these large powers. Consider that if, say x^{200} is larger than y^{300}, then it must be true that x^2 is larger than y^3 since $x^{200} = (x^2)^{100}$ and $y^{300} = (y^3)^{100}$. We can write each expression with a common power, and then consider the expressions without that common power—effectively taking the 100th root of both. This idea applies to the inequality in this problem. We want n^{24} to be larger than 14^{16}. Since $24 = 3 \cdot 8$ and $16 = 2 \cdot 8$, we can take the 8th root of both, and we have that $n^3 > 14^2$. Since $14^2 = 196$, we seek the least positive integer for which $n^3 > 196$. This is much easier, and we can see that $6^3 = 216 > 196$ but $5^3 = 125 < 196$. Therefore, $n = 6$ is the least integer for which $n^{24} > 14^{16}$. Thus, the answer is **A**. ◆

Exercises

1. The solution of the rational inequality $\dfrac{2}{1-x} > \dfrac{1}{2}$ is [AASU 2004]

   (A) $(-3, \infty)$ **(B)** $(-3, -1)$ **(C)** $(-3, -1]$
   (D) $[-3, 1)$ **(E)** None of these

2. Find the solution set for the inequality $x|x| > \dfrac{1}{x}$. [GAC 2009, Varsity]

   (A) $\left(-\infty, -\dfrac{1}{2}\right) \cup (2, \infty)$ **(B)** $(-1, 0) \cup (1, \infty)$ **(C)** $\left(0, -\dfrac{1}{2}\right) \cup (1, \infty)$
   (D) $\left(0, -\dfrac{1}{2}\right) \cup (2, \infty)$ **(E)** None of these

3. Find the largest integer that satisfies the quadratic inequality $x^2 - 47x + 90 < 0$. [GSW 2015]

   (A) 3 **(B)** 12 **(C)** 15 **(D)** 44 **(E)** 89

Compound Inequalities

A *compound inequality* is two inequalities written as one. An example of such compound inequalities is $-3 < 4x - 7 < 9$. This is actually two inequalities that must be satisfied simultaneously. The two inequalities in $-3 < 4x - 7 < 9$ are

$$-3 < 4x - 7 \qquad \text{and} \qquad 4x - 7 < 9.$$

Since the expression $4x - 7$ is linear, we can solve this compound inequality by manipulating all *three* sides at the same time. For instance,

$$\begin{gathered} -3 < 4x - 7 < 9 \\ 4 < 4x < 16 \\ 1 < x < 4 \end{gathered}$$

where we added 7 and then divided by 4 to all three sides.

If the expression is not linear, then we would need to solve the compound inequality as two separate inequalities. The overlap of each of the two solution sets we obtain is the solution set of the original compound inequality. The next example describes this.

Example 4. If $1 < \dfrac{1}{3-2x} < 3$, then [Cobb 2002, Varsity]

(A) $-\dfrac{16}{3} < x < -\dfrac{4}{3}$ **(B)** $-\dfrac{4}{3} < x < 1$ **(C)** $-\dfrac{3}{4} < x < -\dfrac{3}{16}$
(D) $\dfrac{3}{4} < x < 3$ **(E)** $1 < x < \dfrac{4}{3}$

Solution. We solve the two inequalities $1 < 1/(3-2x)$ and $1/(3-2x) < 3$. We start with the first one. We get

$$\begin{aligned} \frac{1}{3-2x} &> 1 \\ \frac{1}{3-2x} - 1 &> 0 \\ \frac{1-(3-2x)}{3-2x} &> 0 \\ \frac{2x-2}{3-2x} &> 0. \end{aligned}$$

Therefore the critical numbers here are $x = 1$ and $x = 3/2$. We find that the expression $(2x-2)/(3-2x)$ is positive for $1 < x < 3/2$. Next, we solve the second one:

$$\begin{aligned} \frac{1}{3-2x} &< 3 \\ \frac{1}{3-2x} - 3 &< 0 \\ \frac{1-3(3-2x)}{3-2x} &< 0 \\ \frac{6x-8}{3-2x} &< 0. \end{aligned}$$

The critical numbers here are $x = 4/3$ and $x = 3/2$. We find that this is solved for $x < 4/3$ or for $x > 3/2$. These two solution sets overlap with the solution set from the first inequality, and the overlap is $1 < x < 4/3$. This is the solution to the compound inequality. Thus, the answer is **E**. ◆

Many absolute value inequalities are compound inequalities. Something as seemingly simple as $|x-4| < 5$ is actually the compound inequality $-5 < x-4 < 5$.

Example 5. Find the solution set of the inequality $|x-4| < |6-x|$. [Cobb 1993, Varsity]

(A) $(-\infty, 4)$ **(B)** $(-\infty, 5)$ **(C)** $[0, 5]$ **(D)** $(0, \infty)$ **(E)** $(2, 6)$

Solution. We have two possible inequalities: $x-4 < 6-x$ or $x-4 < -(6-x) = x-6$. The first inequality leads to $2x < 10$ so $x < 5$. The second leads to $-4 < -6$ which is false. Hence, the only viable solution set is $(-\infty, 5)$. Thus, the answer is **B**. ◆

Exercises

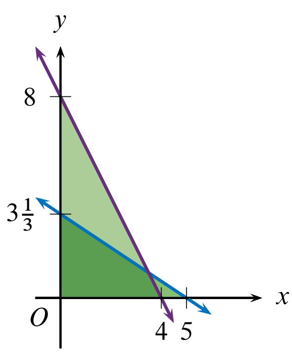
4. Find the set of values satisfying the inequality [Cobb 1992, Varsity]

$$\left|\frac{10-x}{3}\right| < 2.$$

(A) $-16 < x < -4$ **(B)** $-16 < x < 4$ **(C)** $4 < x < 16$
(D) $x < 16$ **(E)** None of these

Linear Programming

Suppose we have a linear equation for which we want a maximum (or minimum) value, subject to certain linear constraints. Perhaps we want to maximize the profit on the manufacture of two models of a product, say model A and model B. The profit on model A is \$34 and on model B is \$23. Suppose the number produced of model A is x and of model B is y. Then the profit P is $p = 34x + 23y$. The process involves two interns, M and N, who are available for this kind of work 100 and 80 minutes per day, respectively. Intern M assembles model A in 20 minutes and model B in 30 minutes. Intern N paints model A in 20 minutes and model B in 10 minutes. These time constraints lead to the inequalities $20x + 30y \le 100$ and $20x + 10y \le 80$. Clearly, we need the number produced to be nonnegative, so $x \ge 0$ and $y \ge 0$.



This kind of problem is called a *linear programming* problem. We want to maximize $p = 34x + 23y$, subject to the *constraints*

$$20x + 30y \le 100, \quad 20x + 10y \le 80, \quad x, y \ge 0.$$

We solve this by graphing the inequalities. Writing them as

$$y \le \frac{10}{3} - \frac{2}{3}x \quad \text{and} \quad y \le 8 - 2x,$$

and noting that since x and y must be nonnegative so we only need the first quadrant, we have the graph shown.

The overlap of the two regions is called the *feasible region* because all points in the region satisfy the constraints. However, there are infinitely many points in the region! We cannot test them all, but the following theorem, whose proof involves concepts from linear algebra, tells us where to look!

Theorem 9: The Corner Point Principle

If a linear programming problem has a unique optimal solution, it must occur at a corner point of the feasible region.

Although the proof is beyond the scope of this book, we can reason by analogy: We can imagine the feasible region as a tray, tilted to one side. Place a marble in the tray. The marble will roll toward the corner closest to the ground, coming to rest at that corner. This is the optimal point.

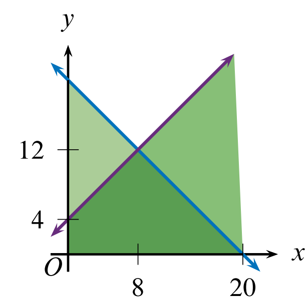
For our problem, we want the maximum of $P = 34x + 23y$, so we only need to consider the corner points farthest from the origin. These points are the intercepts and the intersection. The intercepts are $(4, 0)$, and $(0, 10/3)$. The intersection is found by solving $10/3 - 2x/3 = 8 - 2x$. We get $10 - 2x = 24 - 6x$ so that $4x = 14$. Hence, the intersection is $(7/2, 1)$.

Finally, we calculate P at each point: $P(4, 0) = 4 \cdot 34 = 136$, $P(0, 10/3) = 10/3 \cdot 23 = 230/3 \approx 76.67$, and $P(7/2, 1) = 4 \cdot 7/2 + 34 = 48$. Thus, we find that the maximum profit is \$136. And also notice we have found that to make the most money, we should not produce any model B and only produce model A!

Example 6. If x and y are real numbers with $x + y \le 20$ and $x - y \ge -4$, then the maximum possible value of $x + 2y$ is [AASU 2004]

(A) 8 **(B)** 20 **(C)** 24 **(D)** 32 **(E)** 40

Solution. Since we want the maximum value of $x + 2y$, we may only consider positive values of x and y. Writing the inequalities as $y \le 20-x$ and $y \le x+4$, we can graph them as shown. The shaded regions overlap and this quadrilateral is the feasible region.



The corner points lead to the maximum value. Two of the corner points are the intercepts of the lines $y = 20 - x$ and $y = x + 4$; these are $(20, 0)$ and $(0, 4)$. The other corner point is the intersection of the two lines. Solving $20-x = x+4$ gives us the point $(8, 12)$. Let $z = x+2y$. Then $z(20, 0) = 20$, $z(0, 4) = 2 \cdot 4 = 8$, and $z(8, 12) = 8 + 2 \cdot 12 = 32$. The maximum value is therefore 32. Thus, the answer is **D**. ◆

The Triangle Inequality

Some important results can be obtained from the simple fact that, for all real a, $a \le |a|$. Using this fact, we can derive the following interesting property. Suppose that b is also a real number. Then $a \le |a|$ and $b \le |b|$. Thus,

$$ab \le |a||b|.$$

Multiplying both sides by 2 gives us

$$2ab \le 2|a||b|.$$

Adding a^2 and b^2 to both sides we get

$$a^2 + 2ab + b^2 \le a^2 + 2|a||b| + b^2.$$

However, $a^2 = |a|^2$ for all real a, so that we may write

$$a^2 + 2ab + b^2 \le |a|^2 + 2|a||b| + |b|^2.$$

Now factoring both sides gives us

$$(a + b)^2 \le (|a| + |b|)^2$$
$$|a + b|^2 \le (|a| + |b|)^2$$
$$|a + b| \le |a| + |b|.$$

This result is known as the *triangle inequality*.

This is called the triangle inequality because side lengths of a triangle are always positive and the sum of any two must be greater than the third.

Arithmetic Mean-Geometric Mean

One important result can be obtained from another simple fact that for all non-negative real numbers a and b, we have $(a - b)^2 \ge 0$. Using this fact, we can derive the following useful inequality. Expand the left side to get

$$a^2 - 2ab + b^2 \ge 0.$$

Then add $4ab$ to both sides to get

$$a^2 + 2ab + b^2 \geq 4ab.$$

Now factoring gives us

$$\begin{aligned}(a+b)^2 &\geq 4ab\\ a+b &\geq 2\sqrt{ab}\\ \frac{a+b}{2} &\geq \sqrt{ab}.\end{aligned}$$

This result is known as the *arithmetic mean-geometric mean inequality*. This is also known by the acronym AMGM.

The AMGM inequality can be generalized to three nonnegative real numbers:

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}.$$

Indeed, we have a similar statement for n nonnegative reals:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Example 7. Let $a, b \in \mathbb{R}$ where $a > b > 0$. What is the minimum value that the expression below can equal? [North Fulton 2017, Varsity]

$$2a + \frac{1}{2(a-b)^2 b}$$

(A) 3 **(B)** $\frac{16}{5}$ **(C)** 4 **(D)** $\frac{14}{3}$ **(E)** None of these

Solution. Asking for the minimum value of the sum of two terms is a clue that AMGM is involved. However, we do not know what to do with the difference $a - b$. We can handle this by letting $x = a - b$. Then $a = b + x$. Then the expression becomes

$$2a + \frac{1}{2(a-b)^2 b} = 2(b+x) + \frac{1}{2x^2 b} = x + x + 2b + \frac{1}{2bx^2}.$$

We now use AMGM with four terms: x, x, $2b$, and $1/(2bx^2)$. This yields the inequality

$$x + x + 2b + \frac{1}{2bx^2} \geq 4\sqrt[4]{x \cdot x \cdot 2b \cdot \frac{1}{2bx^2}} = 4\sqrt[4]{1} = 4.$$

Therefore,

$$2a + \frac{1}{2(a-b)^2 b} = x + x + 2b + \frac{1}{2bx^2} \geq 4.$$

So we have proven that the expression is always greater than or equal to 4, but we do not know for sure if it can be *equal to* 4 unless we can find values of x and b that make this expression evaluate to 4 exactly. Luckily, we can: if we let $x = 1$ and $b = 1/2$ (which implies $a = 3/2$), then we do indeed attain the minimum value 4. Thus, the answer is **C**. ◆

Inequality Problems, JV Level

5. If x and y are nonzero integers and $x > y$, which of the following *must* be positive? [AASU 2011]

   (A) xy **(B)** $x + y$ **(C)** $x - y$ **(D)** x/y **(E)** $2x + y$

6. If $0.003 \leq x \leq 0.01$ and $0.001 \leq y \leq 0.1$, then the largest possible value for x/y is [AASU 1988]

   (A) 0.01 **(B)** 0.03 **(C)** 0.1 **(D)** 3 **(E)** 10

7. Which of the following statements are true? [AASU 2011]

   I. $-1 < x < 3$ implies $0 < x^2 < 9$.
   II. $-1 < x < 3$ implies $-1 < x^2 < 9$.
   III. $-1 < x < 3$ implies $1 < x^2 < 9$.

   (A) I only **(B)** II only **(C)** III only
   (D) I and II only **(E)** I, II, and III

8. Which of the following is the least positive integer n for which $n^{12} > 13^8$? [GSW 2011]

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 9

9. Aaron washes cars in the neighborhood and spends \$215 on supplies. He charges \$15 per car. What is the minimum number of cars that must be washed to make a profit? [Luella 2012, JV]

   (A) 13 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** None of these

10. Which of the following is not a solution of $2x - 3y < 7$? [GSW 1997]

    (A) $(-2, -2)$ **(B)** $(-1, 1)$ **(C)** $(0, 2)$ **(D)** $(3, 0)$ **(E)** $(5, -1)$

11. If $0 \leq x + y \leq 21$ and $y \geq 3$, the largest possible value for x is [AASU 1992]

    (A) -24 **(B)** -18 **(C)** 18 **(D)** 24 **(E)** None of these

12. Solve the following: $x^3 + 2x^2 > 8x$. [Luella 2015, JV]

    (A) $x < -4$ **(B)** $x < -4$ or $x > 2$ **(C)** $-4 < x < 0$ or $x > 2$
    (D) $x > 2$ **(E)** None of these